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OPERATOR THEORY ON SPECTRUM OF DISCRETE LAPLACE-BELTRAMI OPERATOR RIEMANNIAN METRIC

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ABSTRACT

In this paper is In this paper some fundamental theorems , definitions in Laplace – Beltrami operator Riemannian geometry to pervious of differentiable manifolds which are used in an essential way in basic concepts of Spectrum of Discrete , bounded Riemannian geometry, we study the defections, examples of the problem of differentially projection mapping parameterization system

INTRODUCTION

Laplace – Beltrami operator plays a fundamental role in Riemannian geometric .In real applications, smooth metric surface is usually represented as triangulated mesh the manifold heat kernel is estimated from the discrete Laplace operator- Discrete Laplace – Beltrami operators on triangulated surface meshes span the entire spectrum of geometry processing applications including mesh parameterization segmentation. The Riemannian manifold with boundary, in the Euclidean domain the interior geometry is given, flat and trivial, and the interesting phenomena come from the shape of the boundary, Riemannian manifolds have no boundary, and the geometric phenomena are those of the interior. The present paper is an introduction, so we have to refrain from saying too must. To any compact Riemannian manifold (M,g) is boundary we associate second- order (P.D.E) , the Laplace operator Δ is defined by : $\Delta(f)=-div(grad f)$ for $f\in L^2(M,g)$. We also sometimes write Δ_g for Δ if we want to emphasize which metric the Laplace operator is associated with the set of eigenvalues of Δ is called the spectrum of Δ or of M which we will write as space Δ or space (M,g) they form a discrete sequence $0=\lambda_0\le\lambda_1\le....\le\lambda_n$ for simplicity For example, we will mainly consider compact Riemannian manifolds . The manifolds to investigated which are manifolds of systems of differential polynomials in a single unknown, possess a degree of analogy to bounded sets of numbers. They are manifolds which may be said (not to contain infinity as a solution) U_α where each set U_α is homeoomorphic, via some homeomorphism h_α to an open subset of Euclidean space R^n , let M be a topological space , a chart in M consists of an open subset $U\subset M$ and a homeomorphism h of U onto an open subset of R^m , a C^r atlas on M is a collection (U_α, h_α) of charts such that the U_α cover M and h_B, h_α^{-1} the differentiable vector fields on a differentiable manifold.Tangent space as defined tangent space to level surface γ be a curve is in R^n, $\gamma: t\to(\gamma^1(t),\gamma^2(t),....,\gamma^n(t))$a curve can be described as vector valued function converse a vector valued function given curve , the tangent line at the point.

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2.1 Basic on Laplacian Riemannian Manifold

Definition 2.1.1

A topological manifold M of dimension n , is a topological space with the following properties:

(a) M is a Hausdorff space . For ever pair of points $p,g \in M$, there are disjoint open subsets $U,V \subset M$ such that $p \in U$ and $g \in V$.

(b) M is second countable . There exists accountable basis for the topology of M . (c) M is locally Euclidean of dimension n Every point of M has a neighborhood that is homeomorphic to an open subset of R^n .

Definition 2.1.2

A coordinate chart or just a chart on a topological $n-$manifold M is a pair (U,φ) , Where U is an open subset of M and $\varphi : U \to \widetilde{U}$ is a homeomorphism from U to an open subset $\widetilde{U} = \varphi(U) \subset R^n$.

Examples 2.1.3

Let S^n denote the (unit) $n-$sphere, which is the set of unit vectors in R^{n+1}: $S^n = \{x \in R^{n+1} : |x| = 1\}$ with the subspace topology, S^n is a topological $n-$manifold.

Definition 2.1.4 [Projective spaces]

The $n-$dimensional real (complex) projective space, denoted by $P_n(R)$ *or* $P_n(C))$, is defined as the set of 1-dimensional linear subspace of R^{n+1} *or* $C^{n+1})$, $P_n(R)$ *or* $P_n(C)$ is a topological manifold.

Definition 2.1.5

For any positive integer n , the $n-$ torus is the product space $T^n = (S^1 \times ... \times S^1)$ .It is an $n-$ dimensional topological manifold. (The 2-torus is usually called simply the torus).

Definition2.1.6

The boundary of a line segment is the two end points; the boundary of a disc is a circle. In general the boundary of an $n-$ manifold is a manifold of dimension $(n-1)$, we denote the boundary of a manifold M as ∂M . The boundary of boundary is always empty, $\partial\partial M = \phi$

Lemma 2.1.7

Every topological manifold has a countable basis of Compact coordinate balls. (B) Every topological manifold is locally compact.

Definitions 2.1.8

Let M be a topological space n -manifold. If $(U,\varphi),(V,\psi)$ are two charts such that $U \cap V \neq \phi$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the transition map from φ to ψ .

Definition 2.1.9

An atlas A is called a smooth atlas if any two charts in A are smoothly compatible with each other. A smooth atlas A on a topological manifold M is maximal if it is not contained in any strictly larger smooth atlas. (This just means that any chart that is smoothly compatible with every chart in A is already in A.

Definition 2.1.10

A smooth structure on a topological manifold M is maximal smooth atlas. (Smooth structure are also called differentiable structure or C^∞ structure by some authors).

Definition 2.1.11

A smooth manifold is a pair $(M$, A), where M is a topological manifold and A is smooth structure on M. When the smooth structure is understood, we omit mention of it and just say M is a smooth manifold.

Definition 2.1.12

Let M be a topological manifold. (i) Every smooth atlases for M is contained in a unique maximal smooth atlas.
(ii) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is smooth atlas.

Definition 2.1.13

Every smooth manifold has a countable basis of pre-compact smooth coordinate balls. For example the General Linear Group The general linear group $GL(n,R)$ is the set of invertible $n\times n$ -matrices with real entries. It is a smooth n^2 -dimensional manifold because it is an open subset of the n^2 - dimensional vector space $M(n,R)$, namely the set where the (continuous) determinant function is nonzero.

Definition 2.1.14

Let M be a smooth manifold and let p be a point of M. A linear map $X : C^{\infty}(M) \to R$ is called a derivation at p if it satisfies :

(1) $$X(fg) = f(p)Xg + g(p)Xf$$

for all $f, g \in C^{\infty}(M)$. The set of all derivation of $C^{\infty}(M)$ at p is vector space called the tangent space to M at p, and is denoted by [T_pM]. An element of T_pM is called a tangent vector at p.

Lemma 2.1.15

Let M be a smooth manifold, and suppose $p \in M$ and $X \in T_pM$. If f is a const and function, then $Xf = 0$. If $f(p) = g(p) = 0$, then $X(fp) = 0$.

Definition2.1.16

If γ is a smooth curve (a continuous map $\gamma : J \to M$,where $J \subset R$ is an interval) in a smooth manifold M, we define the tangent vector to γ at $t_\circ \in J$ to be the vector $\gamma'(t_\circ) = \gamma_*\left(\frac{d}{dt}|_{t_\circ}\right) \in T_{\gamma(t_\circ)}M$, where $d/dt|_{t_\circ}$ is the standard coordinate basis for $T_{t_\circ}R$. Other common notations for the tangent vector to γ are $\left[\gamma^*(t_\circ), \frac{d\gamma}{dt}(t_\circ)\right]$ and $\left[\frac{d\gamma}{dt}|_{t=t_\circ}\right]$. This tangent vector acts on functions by :

(2) $$\gamma'(t_\circ)\, f = \left(\gamma_* \frac{d}{dt}|_{t_\circ}\right) f = \frac{d}{dt}|_{t_\circ} (f \circ \gamma) = \frac{d(f \circ \gamma)}{dt}(t_\circ).$$

Lemma 2.1.17

Let M be a smooth manifold and $p \in M$. Every $X \in (T_pM)$ is the tangent vector to some smooth curve in M.

Definition 2.1.18

A Lie group is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m: G\times G \to G$ and inversion map $m: G \to G$, given by $m(g,h) = gh$, $i(g) = g^{-1}$, are both smooth. If G is a smooth manifold with group structure such that the map $G\times G \to G$ given by $(g,h) \to gh^{-1}$ is smooth, then G is a Lie group. Each of the following manifolds is a lie group with indicated group operation. (a) The general linear group $GL(n,R)$ is the set of invertible $n\times n$ matrices with real entries. It is a group under matrix multiplication, and it is an open sub-manifold of the vector space $M(n,R)$, multiplication is smooth because the matrix entries of A and B. Inversion is smooth because Cramer's rule expresses the entries of A^{-1} as rational functions of the entries of A. The n – torus $T^n = (S^1 \times \ldots \times S^1)$ is an n – dimensional a Belgian group.

Definition 2.1.19 [Lie Brackets]

Let V and W be smooth vector fields on a smooth manifold M. Given a smooth function $f: M \to R$, we can apply V to f and obtain another smooth function Vf, and we can apply W to this function, and obtain yet another smooth function

$(W\,V)f = W(Vf)$. The operation $f \to W\,V\,f$, however, does not in general satisfy the product rule and thus cannot be a vector field, as the following for example shows let $V = \left(\frac{\partial}{\partial x}\right)$ and $W = \left(\frac{\partial}{\partial y}\right)$ on R^n, and let $f(x, y) = x,\ g(x, y) = y$. Then direct computation shows that $V\,W(f\ g) = 1$, while $(f\,V\ W\ g + g\ V\ W\ f) = 0$, so $V\,W$ is not a derivation of $C^\infty(R^2)$. We can also apply the same two vector fields in the opposite order, obtaining a (usually different) function $W\,V\,f$. Applying both of this operators to f and subtraction, we obtain an operator $[V, W] : C^\infty(M) \to C^\infty(M)$, called the Lie bracket of V and W, defined by $[V, W]\ f\ =\ (V\,W)f - (WV)f$. This operation is a vector field. The Smooth of vector Field is Lie bracket of any pair of smooth vector fields is a smooth vector field.

Lemma 2.1.20 [Properties of the Lie Bracket]

The Lie bracket satisfies the following identities for all $V, W, X \in\ (M)$. Bilinearity: $\forall a, b \in R$,

(3) $$[\ aV + bW, X\,] = a\ [V, X] + b\,[W, X] \quad ,$$
$$[\ X, aV + bW\ ] = a\ [X, V] + b\,[X, W].$$

(i) Ant symmetry $[V, W] = -\ [W, V]$.
(ii) Jacobi identity $[V, [\ W, X\ ]] + [\ W, [\ X, V]\ ] + [\ X, [V, W]\ \ ] = 0$. For $f, g \in C^\infty(M)$:

(4) $[f\ V, g\ \ W\ ] = f\,g\,[\ V, W\ ] + (\,f\,V\,g)\ W - (\,g\ W\ \ f)\ \ V$

2.3 Convector Fields

Let V be a finite – dimensional vector space over R and let V^* denote its dual space. Then V^* is the space whose elements are linear functions from V to R , we shall call them Convectors. If $\sigma \in V^*$ then $\sigma : V \to R$ for the any $v \in V$, we denote the value of σ on v by $\sigma(v)$ or by $\langle v, \sigma\rangle$. Addition and multiplication by scalar in V^* are defined by the equations:
(5) $(\sigma_1 + \sigma_2)\ (v)\ =\ \sigma_1(v) + \sigma_2(v)\ ,\ \ (\alpha\sigma)\,(v) = \alpha\ \ (\sigma\ \ (v)\ \ )$
Where $v \in V\ \ , \sigma, \alpha\sigma\ \in V^*$ and $\alpha \in R$.

Proposition 2.3.1 [Convectors]

Let V be a finite- dimensional vector space. If $(E_1,..., E_n)$ is any basis for V ,then the convectors $(\omega^1,...,\omega^n)$ defined by:

(6) $$\omega^i(E_j) = \delta^i_j = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

form a basis for V^* ,called the dual basis to (E_j) .Therefore, $\dim V^* = \dim V$.

Definition 2.3.2 [Convectors on Manifolds]

AC^r – Convector field σ on M , $r \geq 0$, is a function which assigns to each $\beta \in M$ a convector $\sigma_p \in T^*_P(M)$ in such a manner that for any coordinate neighborhood U, ϕ with coordinate frames $E_1,.., E_n$, the functions $\sigma(E_i),\ i = 1,....., n,$ are of class C^r on U . For convenience, "Convector field" will mean C^∞ – convector field.

Remark 2.3.3

It is important to note that a C^r – Convector field σ defines a map $\sigma : \mathcal{H}(\mathrm{M}) \to C^r(M)$, which is not only R – Linear but even $C^r(M)-$ Linear, More precisely, if $f, g \in C^r(M)$ and X and Y are vector fields on M , then $\sigma\ \ (f\ X + g\,Y) = f\sigma\ \ (X) + g\sigma\ \ (Y)$. For these functions are equal at each $p \in M$.

Corollary 2.3.4

Using the notation above let $\sigma = \sum_{i=1}^{n}\alpha_i\ \widetilde{w}^i$ on V, and let $F^*(\sigma) = \sum_{j=1}^{m}\beta_j w^j$ on U ,where ∞_v and β_j are functions on V and U respectively, and $\widetilde{w}^i, w^j$ are the coordinate co frames. Then $F^*(\widetilde{w}^i) = \sum_{j=1}^{m}\frac{\partial y^i}{\partial x^j}w^j$ and $\beta_j = \sum_{i=1}^{n}\frac{\partial y^i}{\partial x^j}\ \ \alpha_i \circ F$. For $i = 1,......, n$

and $j = 1,....., m$. The first formulas give the relation of the bases; the second those of the components. If we apply this directly to a map of an open subset of R^m into R^n, these give for $F^*(dy^i)$ the formula

(7) $$F^*(dy^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} dx^j, \quad i = 1,......, n$$

2.4 The Spectrum of the Laplacian in Riemannian Manifolds

To any compact Riemannian manifold (M,g) is boundary we associate second- order (P.D.E) , the Laplace operator Δ is defined by : $\Delta(f) = -div(grad\, f)$ for $f \in L^2(M, g)$. We also sometimes write Δ_g for Δ if we want to emphasize which metric the Laplace operator is associated with the set of eigenvalues of Δ is called the spectrum of Δ or of M which we will write as space Δ or space (M, g) they form a discrete sequence $0 = \lambda_0 \le \lambda_1 \le .... \le \lambda_n$ for simplicity , we will assume that M is connected . This will for example imply that the smallest eigenvalue λ_0 . Occurs with multiplicity.

Definition2.4.1

If L is aliner operator defined on T_pM , then the spectrum of L is the set of eigenvalues of L .It is denoted by space (L) . We take the Laplace operator Δ defined as $\Delta = -(d\,\delta + \delta d)$, where δ is adjoin of d in spectral geometry we consider the following two equations:

(i) Does the spectrum of M determine the geometry of M.
(ii) Does the geometry of M determine the spectrum of M.

proposition 2.4.2 : [Spectrum Riemannian]

The geometry of Riemannian manifold completely determines the spectrum the metric determines the Laplace operator is spectrum.

Definition 2.4.3: [Sequences be Spectra]

Sequences occur can as the spectra of manifolds a version of this question. Has been answered what finite sequences can occur as the initial part of spectra of manifolds. If M is a closed connected manifold of dimension greater that or Equal to 3, the any p reassigned finite sequence $0 = \lambda_0 \le \lambda_1 \le .... \le \lambda_k$ is Sequence of first $k+1$ eigenvalues of Δ_g for some choice of the metric g on M . In particular, this means that for closed connected manifolds of dimension 3 or Greater, there are no restrictions on the multiplicities of the eigenvalues λ_i for $i \ge 0$. In 2-dimension , there are some restrictions on the multiplicities of the eigenvalues. Let M be a closed connected 2-manifold with Euler characteristic $\chi(M)$, and let m_j be the multiplicity of the $(j-th)$ eigenvalue $(j \ge 0)$ of the laplacian operator associated to a metric on M then : . If M is the unit sphere, then $m_j \le 2_j + 1$

- If M is the real projective plane , then $m_j \le 2_j + 3$
- If M is the torus , then $m_j \le 2_j + 4$
- If M is the klen bottle , then $m_j \le 2_j + 3$
- If , $\chi(M) \le 0$ then $m_j \le 2_j - 2\chi(M) + 3$

[Note]

For finite sequences $0 = \lambda_0 \le \lambda_1 \le .... \le \lambda_k$ however the result by-colin de verdiere holds – even in dimension 2.

2.5 [Estimates on the first Eigenvalue]

The geometry of a manifold affects more that the multiplicities of the eigenvaluees. Here we will focus on bounds on the first non-zero eigenvalue λ_1 imposed by the geometry [the first lower bound is due to lich neowicz].

Theorem 2.5.1: [Ricci Tensor]

Let (M, g) be a closed Riemannian manifold of dimension $n \ge 2$ and let Ric be its Ricci tensor field if.

(8) $\operatorname{Ric}(X,X) \geq (n-1)k \geq 0$

For some constant $k \geq 0$, and for all $X \in T(M)$, then $\lambda_1 \geq nk$.

Theorem 2.5.2

Let (M,g) be a closed Riemannian manifold, if $(X,X) \geq (n-1)k \geq 0$ Ric. For some nonnegative constant k and for all $X \in T(M)$ then.

(9) $$\lambda_1 \geq \frac{(n-1)}{4} + \frac{\pi^2}{D^2(M)}$$

It is in general much easier to given upper bounds on λ_1 that it is give lower bounds. The basic result in this area is a comparison theorem due to a complete Riemannian n- manifold whose Ricci curvature is $\geq (n-1)k, k$ is some const.

Definition 2.5.3

We mentioned a above that a metric g , defines an inner product not just on T_a but also an inner product g^* on T_a^* , with this we can define an inner product on pth exterior power

(10) $T_a^* \Lambda^P : (\alpha_1 \Lambda \alpha_2 \Lambda ......\Lambda \alpha_P, \beta_1 \Lambda \beta_2 \Lambda ....\Lambda \beta_P) = Detg^*(\alpha_i, \beta_i)$

Thus if $dx_1 \Lambda dx_2 \Lambda ....\Lambda dx_n$ defines the orientation $w = \sqrt{\det g_{ij}}\, dx_1 \Lambda ....\Lambda dx_n$
On a compact manifold we can integrate this to obtain total volume – so a metric defines not only length but also volumes, Now take $\alpha \in \Lambda^P T_a^*, \beta \in \Lambda^{n-P} T_a^*$ and define $f_\beta : \Lambda^P T_a^* \to R$, *by* $f_\beta(\alpha) w = \beta \Lambda \alpha$ .But we have an inner product, so any liner map on $\Lambda^P T_a^*$ is of the form $\alpha \to (\alpha, \gamma)$ for some $\gamma \in \Lambda^P T_a^*$ so we have a well –defined liner map $\beta \to \gamma\beta$ form $\Lambda^{n-P} T_a^* to \Lambda^P T_a^*$ satisfying $(\gamma_\beta, \alpha) w = \beta \Lambda \alpha$.

Definition 2.5.4: [Hodge Star Operator]

The Hodge star operator is the linear map $* : \Omega^P(M) \to \Omega^{n-P}(M)$ with the property that at each point.

(11) $(\alpha, \beta) w = \alpha \Lambda * \beta$

Proposition 2.5.5: [Compact Support M manifold]

Let M be an oriented Riemannian manifold with volume for w , and let $\alpha \in \Omega^P(M), \beta \in \Omega^{P-1}(M)$ be forms of compact support then.

(12) $\int_M (d^*\alpha, \beta) w = \int_M (\alpha, d\beta)\, w$

Definition 2.5.6: [Deferential Laplacian on p-forms]

Let M be an oriented Riemannian manifold, then the Laplacian on p-forms is the deferential operator $\Delta : \Omega^P(M) \to \Omega^P(M)$ defined by.

(13) $\Delta : dd^* + d^* d$

Definition 2.5.7: [Starting Point]

A differential form $\alpha \in \Omega^P(M)$ is harmonic if $\Delta\alpha = 0$ .On a compact manifold harmonic ply a important role, which there is no time to explore in this course. Here is the starting point.

Definition 2.5.8:[Harmonic and de Rham Manifold]

Let M be a compact oriented Riemannian manifold then.

(i). a p-form is harmonic if and only if $d\alpha = 0$ and $d^{*}\alpha = 0$
(ii) In each de Rham cohomology class there is at most one harmonic from

Theorem 2.5.9: [Ricci Curvature]

If M is a compact n-manifold with Ricci curvature $\geq (n-1)(-k)$ $, k \geq 0$, then $\lambda_1 \leq \frac{(n-1)^2}{4}k + \frac{c^2}{D^2(M)}$

Where c_2 is positive constant depending only on N.

2.6 [Geometric Implications of The spectrum]

The spectrum does not in general determine the geometry of a manifold Neverthless earthiness , some geometric information can be extracted from the spectrum . In what follows , we define a spectral invariant to be any thing that is completely determined by the spectrum.

Definition 2.6.1

Let M be a Riemannian manifold. A heat kernel or alternatively fundamental solution to the heat equation, is a function $k:(0,\infty)\times(M\times)M \to M$. That satisfies $k(t,x,y)$ is c^1 in t and c^2 in x and y, $\frac{\partial k}{\partial t} + \Delta_2(k) = 0$ where Δ_2 is the Laplacian with respect to the second variable.

(14) $$\lim_{t\to 0^{\pm}} \int_M k(t,x,y) f(y)\,dy = f(x)$$

for any compactly supported function f on M .The heat kernel exists and unique for Riemannian manifold , its importance stems from the fact that the solution to the heat equation

(15) $$\frac{\partial u}{\partial t} + \Delta(u) = 0 \ , \ u:[0,\infty]\times M \to R \ .$$

Where Δ is Laplacian with respect to second variable, with initial condition $u(0,x) = f(x)$ is given by:

(16) $$u(t,x) = \int_M k(t,x,y) f(y)\,dy$$

If $\{\lambda_i\}$ in spectrum of M and $\{\zeta_i\}$ are the associated eigenfunctions (normalized so the they form an orthonormal basis of $L^2(M)$ then we can write $k(t,x,y) = \sum_i e^{-\lambda_i t}\zeta_i(x)\zeta_i(y)$

$spec(m_1/G,g) = spec(m_2/G,g)$ from this it clear that the heat trace $Z(t) = \int_M K(t,x,x) = \sum_i e^{-\lambda_i t_i}$ spectral invariant . The heat trace has an asymptotic expansion as $t \to 0^{\pm}$.

(16) $$Z(t) = (4\pi t)^{\dim M/2} \sum_i^{\infty} a_j t^j$$

Where the a_j are integrals over M of universal homogenous polynomials in the curvature and covariant derivatives. The first few of these are : $a_0 = vol(M)$

(17) $$a_1 = \frac{1}{6}\int_M S \ , a_2 = \int_M (5S^2 - 2|Ric|^2 - |Rm|^2$$

Where S is the scalar curvature, Ric : is the Ricci tensor, Rm : is the curvature tensor. The dimension the volume and total scalar curvature are thus completely determined by spectrum. If M is a surface then the Gauss Bonnet theorem implies that the Euler characteristic of M is also a spectral invariant. Amore in depth study of the heat trace can yield more information of dimension $n \leq 6$ and if M has same spectrum as the n-sphere S^n with the standard metric (resp . RP^m) then M M is in fact isometric to S^n (resp. RP^n) more on this can be found.

Definition 2.6.2 [Isospectral Manifolds]

As was alluded to earlier, geometry is not in general a spectral in variant two manifolds are said to be isospectral if they have the same spectrum. Of non isometric isospectral manifolds was found too distinct but isospectral manifolds.

2.7 [Direct Computation of The Spectrum]

The first of those is straightforward: direct computation. it rarely possible to explicitly compute the spectrum of a manifold were actually discovered via this method . Milnor's example mentioned above consists of two isospectral factory-quotients of Euclidean space by lattices of full rank being one of full rank being one of the few examples of Riemannian manifolds whose spectra can be computed explicicitly spherical space forms – quotients of spheres by finite groups of orthogonal transformations acting without fixed points form another class of examples of manifolds is spectral for the Laplaction acting on p-forms for $R \le k$ but not for the Laplaction acting on p-forms for $R \le k+1$ (recall that a lens space is spherical space form where the group is cyclic.

Theorem 2.7.1

Let $m\Gamma_1$ and $m\Gamma_2$ be compact discrete subgroup of a lie group G, and let g be a left invariant metric on G if $m\Gamma_1$ and $m\Gamma_2$ are representation equivalent then $spec(m_1 / G, g) = spec(m_2 / G, g)$

2.8 Intrinsic Ultracontractivily on Bounded Bomains Manifolds

We first consider on R^d let D be aconnected bounded lipschitz domain in $R^{d\,n}(d \ge 1)$. And Δ with laplacian with Dirichlet boundarg conditions on D . It is well Know that the spectrum of Δ is discrete, $\sigma(-\Delta) = (\lambda_i) \ge 1$ with $0 \le \lambda_1 \le \lambda_2 \le .....$, and each λ_i is an eigenvalue with finite multiplicity. Denote by P_t^D The dirichlet heat kernel on D, and $\phi \ge 0$ the first normalized eigenfunction of $-\Delta$ and D it is also well known that D is intrinsically ultracontractive (i.e).

(18) $$\zeta_t = e^{\lambda_i t} \sup_{x,y\in D} \frac{p_i(x,y)}{\phi(x)\,\Psi(y)} \langle \infty, t \rangle$$

Indeed, this is given true for more general domains such as holder domains of order o. The main purpose of this section is to clarify the short time behavior of ζ_t For lipschitz domains. when D is a … domain.

(19) $$\zeta_t \le 1 + C_t^{-(d+2)/2} \ , \ t \ge 0$$

Holds for some constant ≥ 0 this estimate was extended recently to smooth compact Riemannian manifolds (under some additional) geometrical assumptions) our aim is to study similar estimate for less smooth domains D. we shall see that the estimate, holds for $C^{1,\alpha}$ domains for any $\alpha \ge 0$, If D is metrely lipchitzian (i.e) $C^{1,0}$ is no larger true. For instance, for $D = (0,1)^d$ one has $\phi(x) = \prod_{k=1}^{d} P_t^{(0,1)}(x_i, y_i)$, $\phi(x) = \prod_{k=1}^{d} \sin(\pi, x_i)$. and where $\sin(\pi r)$ is the first dirichlet eigenfunction on [0,1]. Thus combining this with (2,33) below for, $D = (0,1)$ we obtain.

(20) $$\frac{1}{2} t^{-3d/2} \le \zeta_t \le C t^{-3d/2} \ , \ t \in (0,1]$$

For some constant $C \ge 0$. A natatural question is there fore whether for lipschitz domain there exists a constant $C \ge 0$ such that:

(21) $$\zeta_t \le 1 + Ct \ , \ t \ge 0$$

We shall see that the answer is no, in general .It is true that $\zeta_t \le 1 + Ct^{-p}$. for some (qualitative) constant of the boundary.

We prove that for any $B \ge 0$, there exists alipschitz (connected) domain D such that $t^B \zeta_t$ is not bounded $t \to 0$ we summarize this as well as the large time behavior and a lower that domain D ia called Lipschitzian if for any $x \in \partial D$. There exist $S \ge 0$ a coordinate system is called (Lipschilzian), $(r,\theta) \in R \times R^{d-1}$, and a Lipschitz function f on R^{d-1} such that x is the origin and.

(22) $$\begin{aligned} B(x,s) \cap D &= B(x,s) \cap \{(r,\theta) : r \ge f(\theta)\} \\ B(x,s) \cap \partial D &= B(x,s) \cap \{(r,s) : r = f(\theta)\} \end{aligned}$$

A Lipschitz domain is called $C^{1,\alpha}$ for some $\alpha \ge 0$, if the corresponding Lipschitz function satisfies.

(23) $$\left| \nabla f(a) - V(b) \right| \le C \left| a - b \right|^{\alpha}$$

for some $C \ge 0$ and for all, $a,b \in R^{d-1}$. In this definition it is required that ≥ 2 , if $d = 1$, D is an open bounded interval.

Theorem 2.8.1

If D is a $C^{1,\alpha}$ - domain for some ≥ 0 , there exists a constant $C \geq 0$, such that.

(24) $\max\left\{1, \frac{1}{C}t\right\}^{-(a+2)/2} \leq \zeta_t \leq 1 + C(\wedge t)e^{-(d+2)}$, $for\,all\,t \geq 0$

For any $B \geq 0$, there exists a bounded Lipschitz domain $D \subset R^2$ such that : $\lim_{t\to 0} Sup t^B \zeta_t = +\infty$. Now let M be a d-dimensional connected Riemannian manifolds and D an open bounded $C^{1,1}$ domain in M. then for any $x \in \partial D$ there exist $S \geq 0$, a local coordinate system in $(r,\theta) \in R \times R^{d-1}$ in $B(x,s)$. (The open geodesic ball at x with radius ,s) and $f \in C_b^1(R^{d-1})$ with bounded second derivatives such that (3,5) holds. For any.

(25) $y = (r,\theta) \in B(x,s) \cap \overline{D}$ define $f(y) = r - f(\theta) \geq 0$

Then $y = (r,\theta) \in B(x,s) \cap \overline{D}$ has bounded seconed order derivatives furthermore there exists aconstant $C \geq 0$ such that.

(26) $p(y) \leq C|(r,\theta) - f(\theta,\theta)| = cF(y)$

Where ρ is the Rieman $\rho \geq \rho_1$, $on\ D$ nian distance to ∂D . This by the partition of unity, there exists a non negative function $\overline{\rho} \in C_b^1(\overline{D})$ with bounded derivative and $\rho\ |_{\partial D} = 0$ such that:

(27) $\rho \geq \rho_1$, $on\ D$

For some constant $1 \geq 0$, since $\overline{D}$ is compact for simplicity we may and do assume that M is compact to.

(28) $L = N\Sigma_{i=1} X_i^2 + X$

Where X is a bounded measurable vector field and $\{X_i\}_{i=1}^N$ are C^1 vector fields we assume that L is elliptic that is.

(29)
$$(f,f) = \sum_i^N (X_i, f)^2 \geq |\nabla f|^2 \ , \ f \in C^1$$
$$\mu(f^2) \leq r\,\mu(|\nabla f|^2) + B(r)\,\mu(\psi)|f|^2, r \geq 0, f, f \in C^1(M)$$

For some constant $2 \geq 0$. Thus under a local coordinate system on has.

(30) $L = \sum_{i=1}^d a_{ij}\partial_i\partial_j + \sum_{i=1}^d b_i\partial_i$

Where $(a_{ij})_{d\times d}$ is continous and strictly positive definite $b_i(1 \leq i \leq d)$ are bounded measurable. The L-diffusion process uniquely exists.

For any $\in D$, Let $(X_t(x))$ be the L-diffusion process starting from x and, $T(x) = \inf\{t \geq 0, X_t(x) \in \partial D\}$ .For all bounded measurable function f on D .To study the (ontrnsic ultracontractivity) of PD_t. We assume that L is symmetric w.r.t a probability measure $\mu(dx)$, $\mu(dx) = 1, D^{v(x)}dx$ where V is abounded.Measurable function and (dx) the Riemannian volume measure by the ellipticity and the sobolev in equality, we know that spectrum of L on D with dirichlet boundary condition is discrete, As in section 1, we let $\lambda_1 \geq \lambda_2$ be the first two dirichlet eigenvalues and $\phi \geq 0$ the normalized first eigenfunction.

Theorem 2.8.2

Let $D \subset M$ be an open bounded $C^{1,1}$ domain and L a symmetric second order elliptic operator for some bounded measurable vector field X and C^1 - vector fields $\{X\}_{i=1}^N$ such that holds. Then for ζ_t defined in with the present λ_1, ϕ and the transition density μ .

2.9 [Intrinsic ultracontractivity On complete Riemannian manifolds]

Let M be complete, connected, non – compact Riemannian manifold of dimension d. Let $L = \Delta + \nabla V$ for some $\in C^2(M)$. Then L generates a unique (Dirichlet) diffusion semi group P_t on M which is symmetric in $L^2(M)$, where $\mu = e^{V(x)}dx$. for dx the

Riemannian volume measure . A ssume that $\lambda_1 = \inf\sigma(-L)$ is an eigenvalue of $-L$. Since M is connected λ_1 has a unique unite eigenfunction $\theta \geq 0$. In order to study the intrinsic ultracontractivity P_t we make use of the following intrinsic super – poincare inquality introuduced.

(31) $\mu(f^2) \leq r\mu(|\nabla f|^2) + \beta(r)\mu(\Psi)|f|^2$, $C^1(M)$ $r \geq 0, f \in C^1(M)$

Which is equivalent to $\varepsilon(f,g) = \mu(\langle \nabla f, \nabla g \rangle)$ let Ric denote the curvature and ricci curvature on M respectively let ρ be the Riemannian distance on M , and simply write $\rho_0 = \rho(0,\cdot)$ for fixed reference point $0 \in M$ .Let k and K be two positive increasing function on $[0,\infty)$ such that :

(32) $\sec \leq -k_0\rho_0$, $Ric \geq -k_0\rho_0$, $\rho_0 \geq 1$

Holds on M. here $\leq -k_0\rho_0$. means that for any $x \in M$ and unit vectors $X, Y \in T_x$ with $\langle X, Y \rangle = 0$, one has $\sec(X,Y) \leq k$ or $(X,Y) \leq -k(\rho_0(x))$ while $Ric \geq -k_0\rho_0$ means that $Ric(X,Y) \geq -k(\rho_0(x)|X|^2)$ for any $x \in M$ and $X \in T_x$ for a positive increasing function h , on $(0,\infty)$ we let

(33) $h^{-1}(r) = \inf\{s \geq 0, h(s)\}$ $r \geq 0$

Theorem 2.9.1

Let M be a cartan –Hadamard manifold with ≥ 2 , and let $L = \Delta$ A assume that (3,15) holds for some positive incrasing functions k with $k(\infty) = \infty$. We have $\sigma_{ess}(L) = \phi$ holds with.

(34) $\beta(r) = \theta r^{\frac{-d}{2}} \exp\left[\theta k^{-1}(\theta/r)\sqrt{k(4+2k^{-1}(\theta)}\right]$

For some constant $\theta \geq 0$

If $k^{-1}(R)\sqrt{k(4+2k^{-1}(R)} \leq CR^{\varepsilon}$, $R \geq 1$

Holds for some constants $C \geq 0$ and $(0,1)$, then P_t in intrinsically ultra contractive with.

If , holds for comfort some $C \geq 0$ and $\varepsilon = 1$, then P_t is intrinsically hyper contractive , if $\geq -k$ for some constant $k \geq 0$, then $\sigma_{ess}(\Delta) \neq 0$ since M is non-compact and complete, this follows from a comparison for the first Dirichlet eignvalue and the donnely $-L_i$ decomposition principle for rhe essential spectrum. $\inf_{ess}(-\Delta) \leq \sup_{x \in M} \lambda_1(B(x,1) \leq \lambda_1(k)$. Where $\lambda_1 B(x.1)$ is first Dirichlet eigenvalue of $-\Delta$ on D and Where $\lambda_1(k)$ is one on unite geodesic ball in the d-dim. parabolic space with Ricci curvature equal to k .Thus the assumption $k(\infty) = \infty$ is also reasonable . Next, we consider the case with drift. To this end, we adopt the following Bakry – Emery curvature $Ric_{m,l}$. Instead of Ric. A ssume that for some ≥ 0, and positive increasing function k on has instead of the second condition in

(35) $Ric_{l,m} = Ric - H_{ess_v} - \frac{\nabla_v \otimes \nabla_v}{m} \geq -k_0\rho_0$

Moreover, let r be a positive increasing function on $(0,\infty]$ such that $L_{\rho_0} \geq \sqrt{r_0}\rho_0$, $\rho_0 \geq 1$

Theorem 2.9.10

Let o be s pole in M such that hold for some increasing positive functions k and $r(\infty) = \infty$, then $\sigma_{ess}(L) = 0$ moreover, assuming.

(36) $\lim_{\rho_0(x) \to \infty} \frac{\sqrt{k(2+2_{\rho_0}(x))}}{\log^+ \mu(B(x.1)} = 0$

Where $B(x,1)$ is the unit geodesic ball at, we have
holds with.

(37) $B(r) = \theta r^{-(m+d+1)/2} \exp\left[\theta r^{-1}(32/r)\sqrt{k(4)}\right], \theta \geq 0$

For there some constant $\theta \geq 0$

If there exist $C \geq 0$ and $\varepsilon \in (0.1)$ such that $r^{-1}(R)\sqrt{k(2+2r^{-1}(R))} \leq CR^{\varepsilon}$, $R \geq 1$ then P_t is intrinsically ultra contractive and (3,18) holds for some constant $C \geq 0$ .If (3,20) holds for some $C \geq 0$ and $\varepsilon = 1$ then P_t is intrinsically hyper contractive.

Example 2.9.11

let M be a Cartan-Hadamard manifold with $Ric \geq -C(\rho_0^{2(-1)})$ For some constants $C \geq 0$ and α >1, let $v = \theta \rho_0$ for some constant $\theta \geq 0$ and $\rho_0 \geq 1$,then $\sigma_{ess}(L) = 0$ and (2.33) holds $C \geq 0$ For some constant $C \geq 0$ consequently. P_t is is intrinsically ultra contractive if and only if $\theta_1, \theta_2 \geq 2$, and when $\left\| P_t^{\theta} \right\|_{L^2(\mu\theta)} \to L_{M\theta}^{\infty} \leq \theta_1 \exp\left[\theta_1 t^{-/-2}\right]$, $t \geq 0$. For some constants $\theta_1, \theta_2 \geq 0$, which is sharp in the sense that constant θ_2 can not be replaced by any positive function $\theta_2(t) \downarrow 0$ as $t \downarrow 0$. P_t is intrinsically hypercontractive if and only if $\theta_1, \theta_2 \geq 2$.

2.10 The Spectral Geometry of operators of Dirac and Laplace Type

We have also given in each a few additional references to relevant. The constraints of space have of necessity forced us to omit many more important references that it was possible to include and we a apologize in a dance for that. We a the following notational conventions, let (M,g) be compact Riemannian manifold of dim. M with boundary ∂M Let Greek indices μ, γ range from 1 to m , and index a local system of coordinates $x = (x^1, \ldots\ldots\ldots, x^m)$ on the interior of M expand the metric in the form $dS^2 = g_{\mu\gamma} dx^{\mu} dx^{\gamma}$ were $g_{\mu\gamma} = (\partial_{x_\mu}, \partial_{x_\gamma})$ and where we a dopt the Einstein convention of summing over repeated indices we let $g^{\mu\gamma}$ be the inverse matrix the Riemannian measure is given by $dx = (dx^1, \ldots\ldots\ldots, dx^m)$ for $g = \sqrt{\det(g_{\mu\gamma})}$ let ∇ be the levi-Civita connection. We expand $\nabla_{\partial x,\gamma}.\partial_{x\mu} = \Gamma_{\gamma\mu}\sigma \partial x_{\sigma}$ .Where $\Gamma_{\gamma\mu}\sigma$ are the m R are may then be given by.

(38) $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ and given by $R(X,Y,Z,W) = g(R(X,Y),Z,W)$

We shall let Latin indices i,j range from 1 to m and index a local orthonormal frame $\{e_1, \ldots\ldots\ldots, e_m\}$ for the components of the curvature tensor scalar curvature τ Are then given by setting $P_{ij} = R_{ijkk}$ *and* $\tau = P_{ij} = R_{ikki}$. We shall often have an auxiliary vector bundle set V and an auxiliary given on V , we use this connection and the " Levi-Civita" connection to covariant differentiation, let dy be the measure of the induced metric on boundary ∂M, we choose a local orthonormal from near the boundary M , so that $\{e_m\}$ is the inward unit normal. We let indices (a,b) range from 1 to m-1 and index the induced local frame $\{e_1, \ldots\ldots, e_{m-1}\}$ for the tangent bundle f the boundary, let $L_{a,b} =_g (\nabla_{e_a} eb, e_m)$ denote the second fundamaental form. We some over indices with the implicit range indicated. Thus the geodesic curvature K_g is given by $K_g = L_{aa}$. We shall let denote multiple tangential covariant differentiation with respect to the " Levi-Civita" connection the boundary the difference between and being of course measured by the fundamental form.

2.11 [The Geometric of Operators of Laplace and Dirac Type]

In this section we shall establish basic definitions discuss operator of Laplace and of Dirac type introduce the De-Rham complex and discuss the Bochner Laplacian and the weitzenboch formula. Let D be a second of smooth sections $C^{\infty}(v)$ of a vector bundle v over space M, expand.

(39) $D = -\left\{ a^{\mu\nu} \partial x_{\mu} \partial x_{\nu} + a^{\sigma} \partial x_{\sigma} + b \right\}$

where coefficient $\left\{ a^{\mu\nu}, a^{\mu}, b \right\}$ are smooth endomorphism's of v, we suppress the fiber indices . We say that D is an operator of Laplace type if A^2 . On $C^{\infty}(v)$ is said to be an operator of Dirac type if A^2 is an operator of laplace operator of Dirac type if and only if the endomorphisms γ^{ν} satisfy the Clifford commutation relations.

(40) $\gamma^{\nu}\gamma^{\mu}+\gamma^{\mu}\gamma^{\nu}=-2g^{\mu\nu}(id)$.

Let A be an operator of Dirac type and let $\zeta=\zeta_{\nu}dx^{\nu}$ be a smooth 1-form on M we let $\gamma(\zeta)=\zeta_{\nu}v^{\nu}$ define a Clifford module structure on V. This is independent of the particular coordinate system chosen. We can always choose a fiber metric on V so that γ is skew adjoint. We can then construct a unitary connection ∇ on V so that $\nabla\gamma=0$ such that a connection is called compatible the endomorphism if ∇ is compatible we expand $A=\gamma\,\nabla^{v}_{\partial x_{v}}+\psi_{A}$, ψ_{A} is tensorial and does not depend on the particular coordinate system chosen it does of course depend on the particular
compatible connection chosen.

Definition 2.11.1 [The De-Rham Complex]

The prototypical example is given by the exterior algebra, let $C^{\infty}(\Lambda^{p}M)$ be the space of smooth p forms. Let $d:C^{\infty}(\Lambda^{p}M)\to C^{\infty}(\Lambda^{p-1}M)$ be exterior differentiation if ζ is cotangent vector, Let $ext(\zeta):w\to\zeta\Lambda w$ denote exterior multiplication and let $\text{int}(\zeta)$ be the Dual, Interior multiplication, $v(\zeta)=ext(\zeta)-\text{int}(\zeta)$ define module on exterior algebra $\Lambda(M)$. Since $d+\delta=v(dx^{\nu})\nabla_{\partial x_{\nu}}$. $d+\delta$ is an operator of Diract type the a associated laplacian.

(41) $\Delta_{m}=(d+\delta)^{2}=\Delta^{0}_{m}\oplus......\oplus\Delta^{p}_{m}\oplus.....\oplus\Delta^{m}_{M}$

decomposes as the Direct sum of operators of laplace type Δ^{p}_{m} on the space of smooth p forms $C^{\infty}(\Lambda^{p}M)$ on has $\Delta^{0}_{M}=-g^{-1}\partial x_{\mu}gg^{\mu\nu}\partial x_{\nu}$ it is possible to write the p-form valued Laplacian in an invariant form. Extend the " Levi-Civita" comection to act on tensors of all types .Let $\widetilde{\Delta}_{M^{w}}=-g^{\mu\nu}w$, $\mu\nu$ define Bochner or reduced Laplacian, let R given the associated action of curvature tensor. The Weitzenbock formula terms of the Buchner Laplacian in the form

(41) $\Delta_{M}=\widetilde{\Delta}_{M^{w}}+\frac{1}{2}\gamma(dx^{\mu})\gamma(dx^{\gamma})R_{\mu\gamma}$

This formalism can be applied more generally.

Lemma 2.11.2: [Spinor Bundle]

Let D be an operator of Laplace type on a Riemannian manifold, there exists a unique connection ∇ on V and there exists a unique endomorphism E of V, so that $D\phi=-\phi_{ii}-E\phi$ if we express D locally in the form $D=\{g^{\mu\gamma}\partial x_{\gamma}\partial x_{\mu}+a^{\mu}\partial x_{\mu}+b\}$ then the connection 1-form w of ∇ and the endomorphism E are given by $w_{\gamma}=\frac{1}{2}(g_{\gamma\mu}a^{\mu}+g^{\sigma E}\Gamma_{\sigma E_{\gamma}}id)$ *and* $E=b-g^{\gamma\mu}(\partial x_{\gamma}w_{\mu}+w_{\gamma}w_{\mu}-w_{\sigma}\Gamma^{\sigma}{}_{\gamma\mu})$

Let V be equipped with an auxiliary fiber metric, then D is self-adjoin if and only if ∇ is unitary and E is self-adjoin we note if D is the Spinor bundle and the Lichnerowicz formula with our sign convention that $E=-\frac{1}{4}J(id)$ where J is the scalar curvature.

Definition 2.11.3 [Heat Trace Asymptotic for closed manifold]

Throughout this section we shall assume that D is an operator of Laplace type on a closed Riemannian manifold (M,g). We shall discuss the L^{2} - spectral resolution if D is self adjoin, define the heat equation introduce the heat trace and the heat trace asymptotic present the leading terms in the heat trace. Asmptotics references for the material of this section and other references will be cited as needed, we suppose that D is self-adjoin there is then a complete spectral resolution of D on $L^{2}(v)$. This means that we can find a complete orthonormal basis $\{\phi_{n}\}$ for $L^{2}(v)$ where the ϕ_{n} are a smooth sections to V which satisfy the equation $D\phi_{n}=\lambda_{n}\phi_{n}$.

Definition2.11.4

Let V be a vector space and $\varphi\in V$ are tensors. The product of φ and ψ, denoted $\varphi\otimes\psi$ is a tensor of order $r+s$ defined by $\varphi\otimes\psi(v_{1},...v_{r},...v_{r+1},...v_{r+s})=\varphi(v_{1},....,v_{r})\psi(v_{r+1},....,v_{r+s})$. The right hand side is the product of the values of φ and ψ. The product defines a mapping $(\varphi,\psi)\to\varphi\otimes\psi$ of $\mathrm{x}^{r}(V)\to{}^{r+s}(V)$.

Theorem 2.11.5

The product ${}^{r}(V) \circ {}^{r}(V) \rightarrow {}^{r+s}(V)$ just defined is bilinear and associative. If $\omega^1,....,\omega^n$ is a basis of.

(42) $\quad V^* = \varphi_1 \wedge \varphi_2 \wedge .... \wedge \varphi_k = \dfrac{(r_1 + r_2 + ... + r_k)!}{r_1!\, r_2! ... r_k!}$

then $\{(\omega^{i_1} \otimes .... \otimes \omega^{i_r})/(1 \le i_1,...,i_r \le n)\}$ is a basis of $(V)^r$. Finally $F_* : W \rightarrow V$ is linear, then $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$.

Proof

Each statement is proved by straightforward computation. To say that $\otimes$ is bilinear means that if α, β are numbers $\varphi_1, \varphi_2, \in (V)^r$ and $\psi \in {}^{r}(V)$, then $(\alpha\varphi_1 + \beta\varphi_2) \otimes \psi = \alpha(\varphi_1 \otimes \psi) + \beta(\varphi_2 \otimes \psi)$. Similarly for the second variable. This is checked by evaluating each Side on $r + s$ vectors of V ; in fact basis vectors suffice because of linearity Associatively, $(\varphi \otimes \psi) \otimes \theta = \varphi \otimes (\psi \otimes \theta)$, is similarly verified the products on both sides being defined in the natural way. This allows us to drop the parentheses. To see that $\omega^{i_1} \otimes ..... \otimes \omega^{i_r}$ from a basis it is sufficient to note that if $e_1,..., e_n$ is the basis of V dual to $\omega^1,....,\omega^n$, then the tensor $\Omega^{i_1,...,i_r}$ previously defined is exactly $\omega^{i_1} \otimes ..... \otimes \omega^{i_r}$ .This follows from the two definitions:

(43) $\Omega^{i_1...i_r}(e_{j_1},...,e_{j_r}) = \begin{cases} 0 & if\,(i_1,...,i_r) \neq (j_1,...,j_r) \\ 1 & if\,(i_1,...,i_r) = (j_1,...,j_r) \end{cases}$,

and $(\omega^{i_1} \otimes .... \otimes \omega^{i_r})\,(e_{j_1},...,e_{j_r}) = \omega^{i_1}(e_{j_1})\omega^{i_2}(e_{j_2}),...,\omega^{i_r}(e_{j_r}) = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}...\delta^{i_r}_{j_r}$,which show that both tensors have the same values on any (ordered) set of r basis vectors and are thus equal. Finally, given $F_* : W \rightarrow V$, if $w_1,.., w_{r+s} \in W$, then $(F^*(\varphi \otimes \psi))\,(w_1,...,w_{r+s}) = \varphi \otimes \psi\ (F_*(w_1),...,F_*(w_{r+s}))\ \ \varphi(F_*(w_1),..., F_*(w_r))\ \psi(F_*(w_{r+1}),...., F_*(w_{r+s})) = (F^*\varphi) \otimes (F^*\psi)(w_1,.....,w_{r+s})$.

Remark 2.11.7

Let α_p be an element of Λ^p α_p , β_p an element of Λ^q. Then $\alpha_p \wedge \beta_q = (-1)^{pq}\beta_q \wedge \alpha_p$. Hence odd forms ant commute and the wedge product of identical 1-forms will always vanish.

Remark 2.11.8 [Exterior Derivative]

The exterior derivative operation, which takes p-forms into $(p+1)$-forms according to the rule :

(44) $C^\infty(\Lambda^0) \xrightarrow{\ \mathrm{d}\ } C^\infty(\Lambda^1)$; $\mathrm{d}(\mathrm{f}(\mathrm{x})) = \dfrac{\partial \mathrm{f}}{\partial \mathrm{x}^\mathrm{i}} dx^i$ $\qquad C^\infty(\Lambda^1) \xrightarrow{\ \mathrm{d}\ } C^\infty(\Lambda^2)$; $\mathrm{d}(\mathrm{f_j}(\mathrm{x})\mathrm{dx}^\mathrm{j}) = \dfrac{\partial \mathrm{f_j}}{\partial \mathrm{x}^\mathrm{i}} dx^i \wedge dx^j$

$C^\infty(\Lambda^2) \xrightarrow{\ \mathrm{d}\ } C^\infty(\Lambda^3)$; $\mathrm{d}(\mathrm{f_{jk}}(\mathrm{x})\mathrm{dx}^\mathrm{j} \wedge \mathrm{dx}^\mathrm{k}) = \dfrac{\partial \mathrm{f_j}}{\partial \mathrm{x}^\mathrm{i}} dx^i \wedge dx^j \wedge dx^k$

Here we have taken the convention that the new differential line element is always inserted before any previously existing wedge products.

Property 2.11.9

An important property of exterior derivative is that it gives zero when applied twice: $d\ dw_p = 0$. This identity follows from the equality of mixed partial derivative, as we can see from the following simple example:

(45) $C^\infty(\Lambda^0) \xrightarrow{\ \mathrm{d}\ } C^\infty(\Lambda^1) \xrightarrow{\ d\ } C^\infty(\Lambda^2)$

$df = \partial_j f dx^j$, $ddf = \partial_i \partial_j f dx^i \wedge dx^j = \dfrac{1}{2}(\partial_i \partial_j f - \partial_j \partial_i f) dx^i \wedge dx^j = 0.$

Remark 2.11.10

(i) The rule for differentiating the wedge product of a p-form α_p and a q-form β_q is $d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q$.

(ii) The exterior derivative anti-commutes with 1-forms.

Examples 2.11.11

Possible p-forms α_p in two-dimensional space are:

$$\alpha_0 = f(x,y)$$
(46) $$\alpha_1 = u(x,y)dx + v(x,y)dy$$
$$\alpha_2 = \phi(x,y)dx \wedge dy.$$

The exterior derivative of line element gives the two-Dimensional curl times the area $d(u(x,y)dx + v(x,y)dy) = (\partial_x v - \partial_y u)dx \wedge dy$.

The three-space p-forms α_p are.

(47)
$$\begin{aligned} &\alpha_0 = f(x) \\ &\alpha_1 = v_1 dx^1 + v_2 dx^2 + v_3 dx^3 \\ &\alpha_2 = w_1 dx^2 \wedge dx^3 + w_2 dx^3 \wedge dx^1 + w_3 dx^1 \wedge dx^2 \\ &\alpha_3 = \phi(x) dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

We see that

(48)
$$\begin{aligned} &\alpha_1 \wedge \alpha_2 = (v_1 w_1 + v_2 w_2 + v_3 w_3) dx^1 \wedge dx^2 \wedge dx^3 \\ &d\alpha_1 = (\varepsilon_{ijk} \partial_j v_k) \frac{1}{2} \varepsilon_{ijm} dx^1 \wedge dx^m \\ &d\alpha_2 = (\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

(Where ε_{ijk} is the totally anti-symmetric tensor in 3-dimensions).

Definition 2.11.12

An alternating covariant tensor field of order r on M will be called an exterior differential form of degree r (or some time simply, r-form).The set $\Lambda^r(M)$ of all such forms is a subspace of $(M)^r$.

Theorem 2.11.13

Let $\Lambda(M)$ denote the vector space over R of all exterior differential forms. Then for $\varphi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$, the formula , $(\varphi \wedge \psi)_P = \varphi_P \wedge \psi_P$ defines an associative product satisfying $\varphi \wedge \psi = (-1)^{rs} \psi \wedge \varphi$. With this product, $\Lambda(M)$ is algebra over R . If $f \in C^\infty(M)$, we also have $f(\varphi\psi) = \varphi \wedge (f\psi) = \varphi \wedge (f\psi)$ If $\omega^1,...,\omega^n$ is a field of co frames on M (or an open set U of M), then the set

(49) $$\left\{ \left(\omega^{i_1} \wedge ... \wedge \omega^{i_r}\right) / \quad \left(1 \le i_1 < i_2 < ... < i_r \le n\right) \right\}$$
is a basis of $\Lambda^r(M)$ *or* $\Lambda(U)$.

Theorem 2.11.15

If $F: M \to N$ is a C^∞ mapping of manifolds, then $F^*: \Lambda(N) \to \Lambda(M)$ is an algebra homomorphism. (We shall call $\Lambda(M)$ the algebra of differential forms or exterior algebra on M).

Definition 2.11.16

An oriented vector space is a vector space plus an equivalence class of allowable bases, choose a basis to determine the orientation those equivalents to it will be called oriented or positively oriented bases or frames. This concept is related to the choice of a basis Ω of $\Lambda^n(V)$.

Lemma 2.11.17

Let $\Omega \ne 0$ be an alternating covariant tensor on V of order, n = dim V and let $e_1,...,e_n$ .be a basis of V. Then for any set of vectors $v_1,...,v_n$, with $v_i = \sum \alpha_i^j e_j$, , we have , $\Omega(v_1,...v_n) = \det(\alpha_i^j)\ \Omega\ (e_1,...,e_n)$.

Proof:

This lemma says that up to a non vanishing scalar multiple Ω is the determinant of the components of its variables. In particular, If $V = V^n$ is the space on n-tuples and $e_1,...,e_n$ is the canonical basis, then $\Omega(v_1,...,v_n)$ is proportional to the determinant whose rows are $v_1,...,v_n$. The proof is a consequence of the definition of determinant. Given Ω and $v_1,...,v_n$, we use the linearity and ant symmetry of Ω to write.

(50) $$\Omega(v_1,...,v_n) = \sum \alpha_1^{j_1}...\alpha_n^{j_n}\,\Omega(e_{j_1},...,e_{j_n}).$$

Since $\Omega(e_{j_1},...,e_{j_n}) = 0$, if two indices are equal, we may write $\Omega\ (v_1,...,v_n) = \sum \operatorname{sgn}\sigma\,(\alpha_1^{\sigma(1)}...\alpha_n^{\sigma(n)})\Omega\ (e_1,...,e_n) = \det(\alpha_i^j)\ \Omega\ (e_1,...,e_n)$.The last equality uses the standard definition of determinant.

Corollary 2.11.18

Note that if $\Omega \neq 0$, then $v_1,...,v_n$ are linearity independent if and only if $\Omega\ (v_1,...,v_n) \neq 0$. Also note that the formula of the lemma can be construed as a formula for change of component of Ω, there is just one component since $\Lambda^n(V) = 1$, when we change from the basis $e_1,...,e_n$ of V to the basis $v_1,...,v_n$.These statements are immediate consequences of the formula in the lemma.

Definition 2.11.19

We shall say that M is orient able if is possible to define a C^∞ $n-$form Ω on M which is not zero at any point, in which case M is said to be oriented by the choice of Ω. A manifold M is orient able if and only if it has a covering $\{U_\alpha, \varphi_\alpha\}$ of coherently oriented coordinate neighborhoods.

Theorem 2.11.20

Let M be any C^∞ Manifold and let $\Lambda(M)$ be the algebra of exterior differential forms on M .Then there exists a unique R-linear map $d_M : \Lambda(M) \to \Lambda(M)$ such that

(i) If $f \in \Lambda^\circ(M) = C^\infty(M)$, then $d_M f = df$, the differential of f.

$\theta \in \Lambda^r(M)$ *and* $\sigma \in \Lambda^s(M)$, *then* $d_M(\theta \wedge \sigma) = d_M\theta \wedge \sigma + (-1)^r \theta \wedge d_M\sigma$ (ii) $d_M^2 = 0$ .This map will commute with restriction to open sets $U \subset M$, that is, $(d_M\theta)_U = d_U\theta_U$, and map $\Lambda^r(M)$ into $\Lambda^{r+1}(M)$.

3.1 Riemannian Manifold

A bilinear form on a vector space V over R is defined to be a map $\phi : V \times V \to R$ that is linear in each variable separately, that is, for $\alpha, \beta \in R$ and $v, v_1, v_2, w, w_1, w_2 \in V$

(51) $\phi(\alpha v_1 + \beta v_2, w) = \alpha\phi(v_1, w) + \beta\phi(v_2, w)$ $\phi(v, \alpha w_1 + \beta, w_2) = \alpha\phi(v_1, w_1) + \beta\phi(v_1 w_2)$. A similar definition may be made for a map ϕ of a pair of vector space $V \times W$ over R. A bilinear form on V are completely determined by their n^2. Values on basis $e_1,....,e_n$ of V. If $\alpha_{ij} = \phi(e_i, e_j), 1 \le i, j \le n$, are given and $v = \sum \lambda^i e_i$, $w = \sum \mu^j e_j$ are any pair of vectors in V, then bilinearity requires that $\phi\ (v, w) = \sum_{i,j=1}^n \alpha_{ij}\ \lambda^i\ \mu^j$ .A bilinear form, or function is called symmetric if $\phi(v, w) = \phi(w, v)$, and skew–symmetric if $\phi(v, w) = -\phi(w, v)$ asymmetric form is called positive definite if $\phi(v, v) \ge 0$ and if equality holds if and only if $v = 0$; in this case we often call ϕ an inner product on V.

Definition 3.1.1

A field ϕ of C^r-bilinear forms, $r \ge 0$, on a manifold M consists of a function assigning to each point P of M, a bilinear form ϕ_P on $T_P(M)$,that is, a bilinear mapping $\phi_P : T_P(M) \times T_P(M) \to R$, such that for any coordinate neighborhood U, φ, the function $\alpha_{ij} = \phi(E_i, E_j)$, defined by ϕ and the coordinate forms $E_1,...,E_n$,are of class C^r. Unless otherwise stated bilinear forms will be C^∞. To simplify notation we usually write $\phi(X_P, Y_P)$ for $\phi_P(X_P, Y_P)$.

Definition 3.1.2

Suppose $F_*: W \to V$ is a linear map of vector spaces and ϕ is A bilinear form on V.Then the formula $(F^*\phi)(v,w) = \phi(F_*(v), F_*(w))$ defines a linear form $F^*\phi$ on W .

Theorem 3.1.3

Let $F: M \to N$ be a C^∞ map and ϕa bilinear form of class C^r on N . Then $F^*\phi$ is a $C^r -$bilinear form on M . If ϕ is symmetric (skew- symmetric), then $F^*\phi$ is symmetric (skew- symmetric).

Proof

The proof parallels those of theorem and we analogously obtain formulas for the components of $F^*\phi$ in terms of those of ϕ we suppose U,ϕ and V,ψ , are coordinate neighborhoods of p and of $F(P)$ with $F(U) \subset V$ .Using the notation of theorem we may write.

(51) $\beta_{ij}(p) = (F^*\phi)_p(E_{ip}, E_{jp}) = \phi(F_*(E_{ip}), F_*(E_{jp}))$.

Applying as before, we have .

$$\beta_{ij}(p) = \sum_{s,t=1}^{n} \frac{\partial y^s}{\partial x^i}\frac{\partial y^t}{\partial x^j}\ \phi\ (\widetilde{E}_{sF(p)}, \widetilde{E}_{tF(p)}).$$

This gives the formula.

(52) $\beta_{ij}(p) = \sum_{s,t=1}^{n} \frac{\partial y^s}{\partial x^i}\frac{\partial y^t}{\partial x^j} \alpha_{st}(\ F(p)\ )\quad ,\quad 1 \le i,j \le m,$

for the matrix of components $(\beta_{ij})$of $F^*\phi$ at p in terms of the matrix (α_{st}) of ϕ at $F(p)$. The functions β_{ij} thus defined are of class C^r at least on U which completes the proof.

Corollary 3.1.4

If F is an immersion and ϕ is a positive definite, symmetric form then $F^*\phi$ is a positive definite, symmetric bilinear form.

Proof:

All that we need to check is that $F^*\phi$ is positive define at each $P \in M$. Let X_P be any vector tangent to M at p. Then .

(53) $F^*\phi\,(X_P, X_P) = \phi\,(F_*(X_P))$, $F_*\,(X_P) \ge 0$ with equality holding only if $F_*(X_P) = 0$. However, since F is an immersion, $F_*(X_P) = 0$ .if and on only if $X_P = 0$.

Definition 3.1.5

A manifold M on which there is defined a field of symmetric, positive definite, bilinear forms ϕ is called a Riemannian manifold and ϕ the Riemannian metric. We shall assume always that ϕ is of class C^∞ .

Dentition 3.1.6 [Rings Riemannian]

a Riemannian manifold , having define vectors and one-form we can define tensor , a tensor of rank $(m\,,n)$ also called $(m\,,n)$ tensor , is defined to be scalar function of m one-forms and v vectors that is linear in all of its argument, if follow at once that scalars tensors of rank (0,0) , for example metric tensor scalar product equation $\widetilde{P}\,(\vec{V}\,) = \left\langle \widetilde{P}\,,\vec{V}\,\right\rangle$ requires a vector and one-form is possible to obtain a scalar from vectors or two one-forms vectors tensor the definition of tensors , any tensor of $(0,2)$ will give a scalar form two vectors and any tensor of rank $(0,2)$ combines two one-forms to given $(0,2)$ tensor field g_x called tensor the g_x^{-1} inverse metric tensor , the metric tensor is a symmetric bilinear scalar function of two vectors that g_x and g_x is returns a scalar called the dot product . $g\,(\vec{V}\,,\vec{W}\,) = \vec{V}\,.\vec{W} = \vec{W}\,.\vec{V} = g\,(\vec{W}\,,\vec{V}\,)$ .Next we introduce one-form is defined as linear scalar function of vector $\widetilde{P}\,(\vec{V}\,)$ is also scalar product $\widetilde{P}\,(\vec{V}\,) = \left\langle \widetilde{P}\,,\widetilde{V}\,\right\rangle$ one-form $\widetilde{p}$ satisfies the following relation.

$$\widetilde{P}\,(a\vec{V} + b\vec{W}\,) = \left\langle P\,, a\vec{V} + b\vec{W}\,\right\rangle = a\left\langle \widetilde{P}\,,\vec{V}\,\right\rangle + b\left\langle \widetilde{P}\,,\vec{W}\,\right\rangle = a\widetilde{P}\,(\vec{V}\,) + b\widetilde{P}\,(\vec{W}\,)$$

and given any two scalars a and b and one-forms $\tilde{P}, \tilde{Q}$ we define the one-form $a\tilde{P} + b\tilde{Q}$ by.

$(a\tilde{P} + b\tilde{Q})(\vec{V}) = \langle a\tilde{P} + b\tilde{Q}, \vec{V} \rangle = a\langle \tilde{P}, \vec{V} \rangle + b\langle \tilde{Q}, \vec{V} \rangle$

$= a\tilde{P}(\vec{V}) + b\tilde{Q}(\vec{V})$

and scalar function one-form we may write $\langle \tilde{P}, \vec{V} \rangle = \tilde{P}(\vec{V}) = \vec{V}(\tilde{P})$ .For example $m = 2$, $n = 0$ and

$T(a\tilde{P} + b\tilde{Q}, c\tilde{R} + d\tilde{S})$

$= acT(\tilde{P}, \tilde{R}) + adT(\tilde{P}, \tilde{S}) + bcT(\tilde{Q}, \tilde{R}) + bdT(\tilde{Q}, \tilde{S})$

tensor of a given rank form a liner algebra mining that a liner combinations of tensor rank (m, n) is also a tensor rank (m, n) , and tensor product of two vectors A and B given a rank $(2,0)$, $T = \vec{A} \otimes \vec{B}$, $T(\tilde{P}, \tilde{Q}) \equiv \vec{A}(\tilde{P}).\vec{B}(\tilde{Q})$ and $\otimes$ to denote the tensor product and non commutative $\vec{A} \otimes \vec{B} \neq \vec{B} \otimes \vec{A}$ and $\vec{B} = c\vec{A}$ for some scalar , we use the symbol $\otimes$ to denote the tensor product of any two tensor e.g $P \otimes T = \tilde{P} \otimes \tilde{A} \otimes \tilde{B}$ is tensor of rank $(2,1)$. The tensor fields in inroad allows one to the tensor algebra $A_R(T_pM)$ the tensor spaces obtained by tensor protects of space R , T_pM and $T^*{}_pM$ using tensor defined on each point $p \in M$ field for example M be n-dimensional manifolds a differentiable tensor $t_p \in A_R(T_pM)$ are same have differentiable components with respect , given by tensor products of bases $\left(\frac{\partial}{\partial x^k} \right)_p \subset T_pM$, $k = 1,....,n$ and $(dx^k)_p \subset T^*{}_pM$ induced by all systems on M .

3.2 Riemannian Manifold on Curvature Bounded

Let M be complete Riemannian manifold with sectional curvature bounded below by a constant $-K^2$ Let $u \in USC(M)$ *and* $v \in LSC(M)$ be tow functions satisfying. $\mu_0 := \sup_{x \in M}[u(x) - v(x)] \leq +\infty$. Assume that u and v are bounded from above and below respectively and there exists a function $W:[0,\infty] \to [0,\infty]$ satisfying $W(l) \succ 0$. when $l \succ 0$. and $W(0+) = 0$ such that $u(x) - u(y) \leq w(d(x,y))$ Then for each $\varepsilon \succ 0$ there exist $x_\varepsilon, y_\varepsilon \in M$, such that $(p_\varepsilon, X_\varepsilon) \in \bar{J}^{2,+}u(x_\varepsilon)$, $(q_\varepsilon, Y_\varepsilon) \in \bar{J}^{2,+}v(y_\varepsilon)$ such that $u(x) - u(y) \geq \mu_0 - \varepsilon$. And such that $d(x_\varepsilon, y_\varepsilon) \prec \varepsilon$, $| p_\varepsilon - q_\varepsilon \circ P_\gamma(l) | \prec \varepsilon$, $X_\varepsilon \leq Y_\varepsilon \circ p_\gamma(l) + \varepsilon P_\gamma(l)$

Where $l = d(x_\varepsilon, y_\varepsilon)$ *and* $p_\gamma(l)$ is the parallel transport along the shortest geodesic connecting x_ε *and* y_ε.We divide the proof into tow

parts. [a]: without loss of generality , we assume that $\mu_0 \geq 0$. Otherwise we replace u by $u - \mu_0 + 1$ for each $\alpha \succ 0$ we take $\hat{x}_\alpha \in M$ such $u(\hat{x}_\alpha) - v(\hat{x}_\alpha) + w\left(\sqrt{\frac{\mu_0}{\alpha}}\right) \geq \mu_0$.

Part[2]: We apply to $\varphi_\alpha(x,y) = \frac{\alpha}{2}d(x,y)^2 + \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, x)^2 + \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, y)^2$. We have for any $\delta \succ 0$ there exist $X_\alpha \in ST^*_{x\beta}(M)$ *and* $Y_\alpha \in ST^*_{y\alpha}(M)$ such that .$(D_x\varphi_\alpha(x_\alpha, y_\alpha), X_\alpha) \in \hat{J}^{2,+}u(x_\alpha)$ and $(-D_y\varphi_\alpha(x_\alpha, y_\alpha), X_\alpha) \in \hat{J}^{2,-}u(y_\alpha)$ and the block diagonal matrix satisfies $-\left(\frac{1}{\delta} + \|\Lambda_\alpha\|\right)I \leq \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leq \Lambda_\alpha + \delta\Lambda_\alpha^2$

Corollary 3.2.1:[Complete Manifolds with Ricci Curvature Bounded]

Let M be complete Riemannian Manifold with Ricci curvature bounded below by a constant $-(n-1)k^2$ and f *a* C^2 function on M bounded from below then for any $\varepsilon \succ 0$ there exist a point $x_\varepsilon \in M$ such that

$f(x_\varepsilon) \leq \inf f + \varepsilon, | \nabla f |(x_\varepsilon) \leq \varepsilon, \Delta f(x_\varepsilon) \geq -\varepsilon$

Proof :

Let $u = \inf f$ *,and* $v = f$ .w can be chose to be a linear function. It is straightforward to verify that all conditions in the theorem are satisfied.

3.1 Discrete Laplace-Beltrami operator metric

Laplace – Beltrami operator plays a fundamental role in Riemannian geometric .In real applications, smooth metric surface is usually represented as triangulated mesh the manifold heat kernel is estimated from the discrete Laplace operator- Discrete Laplace – Beltrami operators on triangulated surface meshes span the entire spectrum of geometry processing applications including mesh parameterization segmentation.

Definition 3.1.1 [Laplace – Beltrami Operator]

Suppose (M,g) is complete Riemannian manifold, g is the Riemannian metric, Δ is Laplace – Beltrami operator . The eigenvalue $\{\lambda_n\}$ and eignfunctions $\{\phi_n\}$ of Δ are $\Delta\phi_n = \lambda_n\phi_n$, where ϕ_n is normalized to be orthonormal in $L^2(M)$, the spectrum is given by $0 = \lambda_0 \le \lambda_1 \le ....\le \lambda_n$ and $\lambda_n \to \infty$ then there is heat kernel $K(x,y,t) \in C^\infty(M \times M \times R^+)$ such that $K(x,y,t) = \sum e^{-\lambda_n t}\phi_n(x)\phi_n(y)$
heat kernel reflects all the information of the Riemannian metric.

Theorem 3.1.2

Let $f:(M,g) \to (M_2,g_2)$ diffeomorphism between two Riemannian manifold , If f is an isometric $K(x,y,t) = K_2(Mf(y),t)f(x)$ $\forall x,y \in M, t \ge 0$ Conversely, if f is subjective map and equation holds then f is an isomety.

Definition 3.1.2 : [Polyhedral Surface]

An Euclidean polyhedral surface is a triple (S,T,d) , S: is a closed surface, T: is a triangulation of S , d : is metric on S , whose restriction to each triangle is isometric to on Euclidean triangle.

Definition 3.1.4 : [Cotangent Edge Weight]

Suppose $\lfloor V_i,V_j \rfloor$ is boundary edge of M and $\lfloor V_i,V_j \rfloor \in \partial M$, Then $\lfloor V_i,V_j \rfloor$ is associated with one triangle $\lfloor V_i,V_j.V_k \rfloor$ the against $\lfloor V_i,V_j \rfloor$ at the vertex V_k is α then the weight of $\lfloor V_i,V_j \rfloor$ is given by $W_{ij} = \frac{1}{2}\cot\alpha$, otherwise if $\lfloor V_i,V_j \rfloor$ is an interior edge the two angles are α,β then the weight is $W_{ij} = \frac{1}{2}(\cot\alpha + \cot\beta)$.

Definition 3.1.5 : [Discrete Heat Kernel]

The discrete heat kernel is defined as, $K(t) = \phi \exp(-\wedge C\ )\ \phi^T$

Definition 3.1.6

Suppose two Euclidean polyhedral surfaces (S,T,d_1) and (S,T,d_2) are give $L_1 = L_2$ if and d_1 and d_2 differ by a scaling .
Suppose two Euclidean polyhedral, surface (S,T,d_1) and (S,T,d_2) are given $K_1(t) = K_2(t)$ $\forall t \ge 0$, if d_1 and d_2 differ by a scaling.

Proof :

Therefore the discrete Laplace metric and the discrete heat kernel mutually determine each other. We fix the connectivity of polyhedral surface (S,T) .Suppose the edge set of (S,T) is sorted as $E = (e_1, e_2, ..., e_m)$ where $m = |E|$, the face set as F and a triangle $\lfloor V_i,V_j.V_k \rfloor \in F$ as $\{i,j,k\} \in F$. We denote an Euclidean polyhedral metric $d \equiv (d_1,d_2,.........,d_m)$ where $d: E \to R^+$ is the edge length function $d_i : d(e_i)$ is the length of edge e_i is $E_d(2) = \{(d_1,d_2d_3)\ |d_i + d_j| \ge d_k\}$, Be the space of all Euclidean triangles parameterized by the edge where $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$. In this work for convene, we use $u = (u_1,u_2,........,u_m)$. To represent the metric, where $u_k = \frac{1}{2}d_k^2$.

Definition 3.1. 7: [Energy]

An Energy $E:\Omega_u \to R$ is defined as $E(u_1,u_2.........,u_m) = \int_{(1,1,.....1)}^{(u_1,u_2,.....,u_m)} \sum_{k+1}^{m} W_k(u)d\mu_k$. Where $W_k\ (u)$ the cotangent weight on the edge e_k determined by the metric μ.

Lemma 3.1.8

Suppose $\Omega \subset R^n$, is an open convex domain in R^n , $E:\Omega \to R$ is a strictly convex function with positive definite Hessian matrix then $\nabla E:\Omega \to R^n$ is a smooth embedding we show that Ω_u is a convex domain in R^m, the energy E is convex . According to gradient of energy.

$\nabla E(d):\Omega \to R^m$, $\nabla E=(u_1,u_2,..........,u_m)\to(w_1,w_2,........,w_m)$ is an embedding Namely the metric determined by the edge weight unique up to a sealing.

Lemma 3.1.9

Suppose an Euclidean triangle is with angles (ϕ_i,ϕ_j,ϕ_k) and edge lengths (d_i,d_j,d_k) Angles are treated as function of the edge lengths $\phi_i(d_i,d_j d_k)$ then $\frac{\partial\phi_i}{\partial d_i}=\frac{d_i}{2A}$, $\frac{\partial\phi_j}{\partial d_j}= -\frac{d}{2A}\cos\phi_k$. Where A is the area of the triangle.

Lemma 3.1.10

In an Euclidean triangle, let $u_i=\frac{1}{2}d_i^2$ and $u_j=\frac{1}{2}d_j^2$, then $\frac{\partial\cot\phi_i}{\partial u_j}=\frac{\partial\cot\phi_j}{\partial u_i}$

Corollary 3.1.11

The differential form $W=\cot\phi_i du_i+\cot\phi_j du_j+\cot\phi_k du_k$. Is a closed 1-form.

Corollary 3.1.112 [Open Surfaces]

The mapping on an Euclidean polyhedral surface with boundaries $\nabla E:\Omega_u \to R^m$, $(u_1,u_2,..........,u_m)\to(w_1,w_2,........,w_m)$ is smooth embedding, it can proven using double covering technique.

3.2 [A Liouville Type Theorem for Complete Riemannian Manifolds]

First we consider the most popular maximum principle, let U be an connected set in an m-dimensional Euclidean space R^m and $\{x^j\}$ a Euclidean coordinate. We denote by L a differential operator defined by.

(3.1) $$L=\Sigma a^{ij}\frac{\partial^2}{\partial x^i\partial x^j}+\Sigma\, b^j\frac{\partial}{\partial x^j}$$

Where a^{ij} and b^j are smooth function on U for any indices . When the matrix a^{ij} is positive definite and symmetric , it is called a second order elliptic differential operator . We assume that L is an elliptic differential operator . The maximum principle is explained as follows.

Defections 3.2.1 [Maximum Harmonic on Riemannian Geometry]

For a smooth function f on U if it satisfies $Lf \ge 0$, and if there exists a point in U at which it attains the maximum , namely , if there exists a point x_0 in U at which $f(x_0)\ge f(x)$, for any point x in M then the function f is constant. In Riemannian Geometry . this property is reformed as follows . Let (M,g) be a Riemannian manifold with the Riemannian metric g , then we denote by Δ the Laplacian associated with the Riemannian metric g a function f is said to sub harmonic or harmonic if satisfies $\Delta f \ge 0$ *or* $\Delta f = 0$

Defection 3.2.2

For a sub harmonic function on f on Riemannian manifold M if there exist a pints in M at which attains the this property is to give a certain condition for a sub harmonic function to be constant , when we give attention to the fact relative t these maximum principles.

Definition 3.2.3Liouvile's

(a)Let f be a sub harmonic function on R^n, if it is bounded then it is constant.(b) Let f be a harmonic functions on R^n , $m\ge 3$. If it is bounded then it is constant . We are interested in Riemannian analogues of Liouvile,s theorem compared with these Last tow theorems we give attention to the fact that there is an essential difference between base manifold . In fact one is compact and the other is complete and an compact , we consider have a family of Riemannian manifold (M,g) at the global situations it suffices to consider a bout the family of complete Riemannian manifold of course , the subclass of compact Riemannian manifolds. (M,g): is complete Riemannian manifold since a compact Riemannian manifold .

Theorem 3.2.4 [Complete Riemannian Manifold]

A let M be complete Riemannian manifold whose Ricci curvature is bounded from below , if C^2 - nonnegative function f satisfies Where Δ denotes the Laplacian on M , then f vanishes identically, the purpose of this theorem is t prove the following (Leadville Type) theorem in a complete Riemannian manifolds similar to theorem in a complete Riemannian manifold similar to give anther proof of (Nishikawas theorem) . In this note main theorem is as follows

3.3 Riemannian Manifold whose Ricci is Bounded

Let M be a complete Riemannian manifold whose Ricci curvature is bounded from blew , if C^2 - nonnegative function f satisfies Where C_0 is any positive constant and n is any real number greater f vanishes identically .

Theorem 3.3.1 [Ricci Riemannian Manifold]

Let M an n-dimensional Riemannian manifold whose Ricci curvature is bonded from below on M , Let G be a C^2 - functions bounded from below on M , then for any $\varepsilon \geq 0$, there exists a point p such that

$|\nabla G(P)| \leq \varepsilon$, $\Delta G(P) - \varepsilon$ *and* $\inf G + \varepsilon \geq G(p)$

Proof :

In this section we prove the theorem stated in introduction first all in order prove theorem , then our theorem is directly obtained as a corollary of this property and hence Nishikawas theorem is also a direct consequence of this (Nishikawas one)

Theorem 3.3.2 [Manifold and Ricci Curvature]

Let M be a complete Riemannian manifold whose Ricci Curvature is bounded from blew , Let F be any formula of the variable F with constant coefficients such that $F(f) = (C_0 f^n + C_1 f^{n-1} + \ldots\ldots\ldots\ldots + C_k f^{n-k}) + C_{k+1}$ Where $n \geq 1$, $1 \geq n-k \geq 0$ and $C_0 \geq C_{k+1}$ if a C^2 - nonnegative function f satisfies . Then we have Where f_1 denotes the super mum f the given function f .

Proof :

From the assumption there exists a positive number a which satisfies $C_{k+1} \leq a^n C_0$ For the constant a given above the function $G(f)$ with respect to 1-variable f is defined by $(f+a)^{\frac{1-n}{2}}$, n is the maximal degree of the f , then it is easily seen that G is the C^2 - function so that it is bounded from appositive by the constant $a^{\frac{1-n}{2}}$ and bounded from below by 0 , By the simple calculating we have

(53) $$\nabla G = -\frac{n-1}{2} G^{\frac{n+1}{n-1}} \nabla f$$

Hence we get by using the above equation $\frac{1-n}{2} G \frac{2n}{n-1} \Delta f = G\Delta G - \frac{n+1}{n-1} |\nabla G|^2$ Since the Ricci curvature is bounded from below by the assumption and the function G defined above satisfies the condition that it is bounded from below , we can apply the theorem to the function G. Given any positive number ε there exist a point P at which it satisfies (3.2) and (3.3) , (3.4) the following relationship at P .

$$\frac{1-n}{2} G(P)^{\frac{2n}{n-1}} \Delta(f) \geq -\varepsilon G(P) - \frac{n+1}{n-1} \varepsilon^2$$

Can be derived , where $G(P)$ denotes $G(f\varphi)$ thus for any convergent sequence e $G_0 = \inf G$, by taking a sub sequence , if necessary because the sequence is bounded and therefore each term $G(P_m)$ of the sequence satisfies equation we have $G(P_m) \to G_0 = \inf G$ and the assumption $n \geq 1$ .An the other hand it follows from (5.2) we have

$$\frac{1-n}{2} G(P_m)^{\frac{2n}{n-1}} \Delta(P_m) \geq -\varepsilon_m G(P_m) - \frac{n+1}{n-1}\varepsilon_m{}^2$$

And the right side of the a above inequality converges to zero because the function G is bounded by choosing the constant a it satisfies $C_{k+1} a^{-n} \leq C_0$, A accordingly there is a positive number δ such that $\frac{1-n}{2} C_{k+1} a^{-n} \leq \delta \leq \frac{n+1}{2} C_0$, C_0 is the constant coefficient of the maximal degree of function F so for a given such that $a\,\delta \geq 0$, we can take a sufficiently large integer m such that

(3.5) $$\frac{1-n}{2} G(P_m)^{\frac{2n}{n-1}} F(f(P_m)) \geq -\delta$$

Where we have used the assumption equation (3.2) of the theorem (3.2.6) and equation (3.4) so this inequality together with the definition of $G(P_m)$ Yield $F(f(P_m)) \leq \frac{2\delta}{n-1}(f+a)\,(P_m)^n$

Remark 3.3.4

Suppose that a nonnegative function f satisfies the condition we can directly yield $\nabla f^{\,n-1} = (n-1) f^{\,n-2} \nabla f$,

(3.6) $$\Delta f^{\,n-1} = (n-1)(n-2)\, f^{n-3}\, \nabla(f\, \nabla f) + (n-1) f^{n-2} \Delta f$$

we define a function h by f^{n-1}, if $n \geq 2$ then it satisfies $\Delta h \geq (n-1)\, C_0 h^2$ Thus concerning the theorem in the case $n \geq 2$ the condition (2.7) is equivalent $1 \leq n \leq 2$ where C_1 is a positive constant.

Definition 3.3.5 :[Hypersuface on Curvature $\geq H_0$]

Let U be an open set in the Riemannian manifold (M, g) then .(a) ∂U has mean curvature $\geq H_0$ in the sense of contact hypersurfaces iff for all $q \in \partial U$ and $\varepsilon \geq 0$ there is an open set D of M with $\bar{D} \subseteq \bar{U}$ and $q \in \partial D$ near q is a C hypersuface of M and at point q, $H_q^{\partial D} \geq H_0 - \varepsilon$.(b) ∂U has mean curvature $\geq H_0$ in the sense contact hypersurface is constant $C_k \geq 0$ so that for all $q \in k$ and $\varepsilon \geq 0$ there is open set D of M with $\bar{D} \subseteq \bar{U}$ and $q \in \partial D$ the of ∂D near q, $H_q^{\partial D} \geq H_0 - \varepsilon$ and also $H_q^{\partial D} \geq -C_{Kg}\, |_{\partial D}$.

Theorem 3.3.6

Let (M, g) be a Riemannian manifold $U_0, U_1 \subset M$ open sets and let H_0 be a constant , assume that .(a) $U_0 \cap U_1 = 0$ (b) ∂U_0 has mean curvature $\geq -H_0$ in the sense of contact hyper surfaces. (c) ∂U_1 has mean curvature $\geq H_0$ in the sense of contact hypersurfaces with a one sided Hessian bound .4. there is a point $P \in \bar{U}_0 \cap \bar{U}$ and a neighborhood N of P that has coordinates $(x^1, x^2, \ldots, x^n)$ cantered at P so that for some $r \geq 0$ the image of these coordinates is the box $(x^1, x^2, \ldots x^n) = |x^i| \leq r$ and there are Lipschitz continues and there are Lipschitz continuous function $U_0, U_1 : \{ (x^1, x^2, \ldots, x^{n-1} : |x^i| \leq r) \}$, $(-r, r)$ so that $U_0 \cap N$ are given by

(54) $$U_0, N = \{(x^1, x^2, \ldots, x^n) : x^n \geq U_0(x^1, x^2, \ldots, x^{n-1}) \} \quad , \quad U_1, N = \{(x^1, x^2, \ldots, x^n) : x^n \geq U_1(x^1, x^2, \ldots, x^{n-1}) \}$$

This implies $U_0 \equiv U_1$ and U_0 is smooth function , therefore $\partial U_0 \cap N = \partial U_1 \cap N$ is a smooth embedded hyper surface with constant mean curvature H_0 (with respect to the outward normal to U_1).

Definition 3.3.7

Let M_1 and M_2 be differentiable manifolds a mapping $\varphi : M_1 \to M_2$ is a differentiable if it is differentiable , objective and its inverse φ^{-1} is diffeomorphism if it is differentiable φ is said to be a local diffeomorphism at $p \in M$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi : U \to V$ is a diffeomorphism , the notion of diffeomorphism is the natural idea of equivalence between differentiable manifolds , its an immediate consequence of the chain rule that if $\varphi : M_1 \to M_2$ is a diffeomorphism then $d\,\varphi : T_p M_1 \to T_{\varphi(p)} M_2$. Is an isomorphism for all $\varphi : M_1 \to M_2$ in particular , the dimensions of M_1 and M_2 are equal a local converse to this fact is the following $d\varphi : T_p M_1 \to T_{\varphi(p)} M_2$ is an isomorphism then φ is a local diffeomorphism at p from an immediate application of inverse function in R^n , for example be given a manifold structure again

A mapping $f^{-1}: M \to N$ in this case the manifolds N and M are said to be homeomorphism , using charts (U, φ) and (V, ψ) for N and M respectively we can give a coordinate expression $\tilde{f}: M \to N$

Example 3.3.8

Let M_1^{-1} and M_2^{-1} be differentiable manifolds and let $\varphi : M_1 \to M_2$ be differentiable mapping for every $p \in M_1$ and for each $v \in T_p M_1$ choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \to M_1$ with $\alpha(M) = p$ and $\alpha'(0) = v$ take $\alpha \circ \beta = \beta$ the mapping $d\varphi_p : T_\varphi(p) M_2$ by given by $d\varphi(v) = \beta'(M)$ is line of α and $\varphi : M_1^{-1} \to M_2^{-1}$ be a differentiable mapping and at $p \in M_1$ be such $d\varphi: T_p M_1 \to T_\varphi M_2$ is an isomorphism then φ is a local homeomorphism

Theorem 3.3.9

The tangent bundle TM has a canonical differentiable structure making it into a smooth 2N-dimensional manifold , where N=dim. The charts identify any $U_p \in U(T_p M) \subseteq (TM)$ for an coordinate neighborhood $U \subseteq M$, with $U \times R^n$ that is hausdorff and second countable is called .The manifold of tangent vectors

Definition 3.310

A smooth vectors fields on manifolds M is map $X : M \to TM$ such that . (a) $X(P) \in T_p M$ for every G (b) in every chart X is expressed as $a_i(\partial / \partial x_i)$ with coefficients $a_i(x)$ smooth functions of the local coordinates x_i .

Theorem 3.3.11 tangent bundle TM

The tangent bundle TM has a canonical differentiable structure making it into a smooth 2N-dimensional manifold, where N=dim. The charts identify any $U_p \in U(T_p M) \subseteq (TM)$ for an coordinate neighborhood $U \subseteq M$, with $U \times R^n$ that is hausdorff and second countable is called (The manifold of tangent vectors).

Definition 3.3.12

A smooth vectors fields on manifolds M is map $X : M \to TM$ such that (a) $X(P) \in T_p M$ for every G (b) in every chart X is expressed as $a_i(\partial / \partial x_i)$ with coefficients $a_i(x)$ smooth functions of the local coordinates x_i .

Conclusion

The paper study Riemannian differenterentiable manifolds is a generalization of locally Euclidean E^n in every point has a neighbored is called a chart homeomorphic , so that many concepts from as differentiability manifolds. We give the basic definitions, theorems and properties of laplaian Riemannian manifolds be comes the specterurm of compact support M and Direct comutation of the spaectrum, and spectral geometry of operators de Rham.

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