Available online at http://www.journalijdr.com

International Journal of
DEVELOPMENT RESEARCH

IJDR

ISSN: 2230-9926

International Journal of Development Research
Vol. 5, Issue, 08, pp. 5222-5228, August, 2015

Full Length Research Article

ON LYAPUNOV-TYPE INEQUALITIES FOR FOURTH ORDER LINEAR DIFFERENTIAL EQUATIONS

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ARTICLE INFO

Article History:
Received 10th May, 2015
Received in revised form
23rd June, 2015
Accepted 12th July, 2015
Published online 31th August, 2015

Key Words:
Fourth Order Linear di¤Erential Equations,
Lyapunov-Type Inequalities,
Third-Point and
Four-Pointboundary Conditions.

ABSTRACT

In this paper, we introduce new Lyapunov-type inequalities for fourth order linear di¤erential equations under the third-point and four-point boundary conditions. The result for four-point boundary conditions improve some existing ones in literature and also, we note that the results for three-point boundary conditions are new.

INTRODUCTION

In this paper, we establish new Lyapunov-type inequalities for the following fourth order linear di¤er- ential equations

$y^{(4)}(t) + p(t)y(t) = 0;$ ……………..…………….. (1.1)

where p(t) is continuous and real-valued function and y(t) is a real nontrivial solution of (1.1) satisfying the four-point boundary conditions

$y(a_1) = y(a_2) = y(a_3) = y(a_4) = 0;$ ………………………….. (1.2)

where and in the sequel $a_1; a_2; a_3; a_4$ 2 R, with $a_1 < a_2 < a_3 < a_4$, and $y(t) = 0$ for 8t 2 $(a_1; a_2)$ [$(a_2; a_3)$ [$(a_3; a_4)$, and three-point boundary conditions

$y(a_1) = y^0(a_1) = y(a_2) = y(a_3) = 0;$ ….….………………..……….. (1.3)

$y(a_1) = y^0(a_2) = y(a_2) = y(a_3) = 0$ ...……………………..…… (1.4)

or

$y(a_1) = y(a_2) = y^0(a_3) = y(a_3) = 0;$ …………………………..…… (1.5)

where and in the sequel $a_1; a_2; a_3$ 2 R, with $a_1 < a_2 < a_3$ and $y(t) = 0$ for 8t 2 $(a_1; a_2)$ [$(a_2; a_3)$.
Consider the following second order linear di¤erential equations

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$y''(t) + p(t)y(t) = 0$: ….………………….………… (1.6)

First of all, Lyapunov [4] established the following inequality

$$\frac{4}{(a_2 - a_1)} \int_{a_1}^{a_2} p(t)dt;$$ ...……………………………….. (1.7)

which is called Lyapunov inequality, for (1.6). If $p(t) \in C([0;\infty);R^+)$ and (1.6) has a real solution $y(t)$ such that

$$y(a_1) = 0 = y(a_2), \quad y(t) = 0, \quad \forall\, t \in (a_1; a_2),$$ ………………………..…….. (1.8)

where $a_1; a_2 \in R$ with $a_1 < a_2$.

Afterwards, this inequality improved to the following inequality,

$$a_2 - a_1 \int_{a_1}^{a_2} (a_2 - t)(t - a_1)p^+(t)dt;$$ ……………………………… (1.9)

which is better than the inequality (1.7), by Hartman [3], where $p^+(t) = \max\{p(t); 0\}$. The importance of the inequality (1.9) is that Hartman [3], obtained a better bound for consecutive zeros of solution of (1.6). Recently, Lyapunov-type inequalities have been obtained for higher order linear differential equations which satisfy n-point boundary conditions, but as far as we know, there are fewer studies with regard four order differential equations that the majority of these studies with regard two-point boundary conditions. Before stating many efforts, it is worth to mention following works. Now, consider the following higher order linear differential equations

$$y^{(n)}(t) + p(t)y(t) = 0$$ ……………………………… (1.10)

and n-point boundary conditions

$$y(a_1) = y(a_2) = \dots = y(a_{n-1}) = y(a_n) = 0;$$ ………...……….………... (1.11)

where $a_1 < a_2 < \dots < a_{n-1} < a_n$ and $y(t) = 0$ for $\forall t \in (a_k; a_{k+1})$, $k = 1; 2; \dots; n-1$.

Theorem A (5, Theorem 2). Let $n \in N$, $2 \le n$ and $p(t) \in C([a_1; a_n]; R)$. If (1.10) has a nontrivial solution $y(t)$ satisfying the boundary conditions (1.11), then the following inequality

$$\frac{(n-2)!n^{n-1}}{(n-1)^{n-2}(a_n - a_1)^{n-1}} \int_{a_1}^{a_n} |p(t)|\, dt$$ ……………………………. (1.12)

holds.

Çakmak [2] corrected the inequality (1.12) as follows. Theorem B (2, Theorem 1). Let $n \in N$, $2 \le n$ and $p(t) \in C([a_1; a_n]; R)$. If (1.10) has a nontrivial solution $y(t)$ satisfying the boundary conditions (1.11) then the following inequality

$$\frac{(n-2)!n^n}{(n-1)^{n-1}(a_n - a_1)^{n-1}} \int_{a_1}^{a_n} |p(t)|\, dt$$ ……………………………… (1.13)

holds.

It is clear that the inequality (1.13) for $n = 4$, reduces to the following inequality

$$\frac{512}{27(a_4 \quad a_1)^3} \int_{a_1}^{a_4} jp(t)j \, dt: \qquad \text{………………………. (1.14)}$$

Some Preliminary Lemmas

In this section, we start by considering the boundary conditions (1.2). It is clear that from the Rolle's theorem there are b_1 2 (a_1 ; a_2), b_2 2 (a_2 ; a_3), and b_3 2 (a_3 ; a_4) such that

$$y^0(b_1) = y^0(b_2) = y^0(b_3) = 0: \qquad \text{………………………….. (2.1)}$$

Similarly, there are c_1 2 (b_1; b_2) and c_2 2 (b_2; b_3) such that

$$y^{00}(c_1) = y^{00}(c_2) = 0: \qquad \text{...………………………. (2.2)}$$

Hence, by using Eq. (1.1) and the conditions (2.2), we have the following equality

$$y^{00}(t) = \frac{1}{2 \quad - \quad 1} \int_{c_1}^{c_2} G(t; s)[y^{(4)}(s)]ds; \qquad \text{……………………….. (2.3)}$$

where

$$G(t; s) = \begin{matrix} (c_2 \quad t)(s \quad c_1); & s \quad t \\ (c_2 \quad s)(t \quad c_1); & t \quad s \end{matrix} : \qquad \text{……………………….. (2.4)}$$

Now, we give …rst lemma and its proof.

Lemma 2.1. Assume that $y(t)$ 2 $C^4([a_1 ; a_4]; R)$ any function satisfying the boundary conditions (1.2) and $y(t) = 0$ for 8t 2 (a_1 ; a_2) [(a_2 ; a_3) [(a_3 ; a_4). Then the following inequality

$$jy(t)j \quad \frac{(a_4 \quad a_1)}{a_4 8} \int_{a_1} (a_4 \quad t)(t \quad a_1) \; ^{(4)}(t) \; dt \qquad \text{……………………... (2.5)}$$

holds.

Proof. Assume that $y(t)$ is a nontrivial solution of (1.1) satisfying the boundary conditions (1.2). Then $y(t)$ satis…es the conditions (2.1) and (2.2). Hence, we have

$$y^{00}(t) = \frac{1}{2 \quad - \quad 1} \int_{c_1}^{c_2} G(t; s)[y^{(4)}(s)]ds; \qquad \text{……………………….. (2.6)}$$

where $G(t; s)$ is given in (2.4). Now, if we take the absolute value of (2.6), then we get

$$|y''(t)| \leq \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} G(t,s) \left| y^{(4)}(s) \right| ds. \qquad \text{………………………. (2.7)}$$

Integrating (2.7) from b_1 to t and then using $\left| \int_{b_1}^{t} y''(u)du \right| \leq \int_{b_1}^{t} |y''(u)|$ du, we get

$$|y'(t)| \leq \frac{1}{c_2 - c_1} \int_{b_1}^{t} \int_{c_1}^{c_2} G(u,s) \left| y^{(4)}(s) \right| dsdu. \qquad \text{………………………. (2.8)}$$

Similarly, we get

$$|y'(t)| \leq \frac{1}{c_2 - c_1} \int_t^{b_3} \int_{c_1}^{c_2} G(u,s) \left| y^{(4)}(s) \right| dsdu.$$
……………………… (2.9)

Adding (2.8) and (2.9), we have

$$\begin{aligned} |y'(t)| &\leq \frac{1}{2(c_2 - c_1)} \int_{b_1}^{b_3} \int_{c_1}^{c_2} G(u,s) \left| y^{(4)}(s) \right| dsdu \\ &= \frac{1}{2(c_2 - c_1)} \int_{c_1}^{c_2} \left| y^{(4)}(s) \right| \left[\int_{b_1}^{b_3} G(u,s) du \right] ds. \end{aligned}$$
…………………….. (2.10)

Now, consider only the integral $\int_{b_1}^{b_3}$ G (u; s)du. Hence, we obtain

$$\begin{aligned} \int_{b_1}^{b_3} G(u,s)du &= (c_2 - s) \int_{b_1}^{s} (u - c_1) du + (s - c_1) \int_{s}^{b_3} (c_2 - u) du \\ &= \left[\frac{(c_2 - s)(s - c_1)^2}{2} + \frac{(c_2 - s)^2 (s - c_1)}{2} \right] \\ &\quad - \left[\frac{(c_2 - s)(b_1 - c_1)^2}{2} + \frac{(c_2 - b_3)^2 (s - c_1)}{2} \right] \\ &\leq \frac{(c_2 - s)(s - c_1)^2}{2} + \frac{(c_2 - s)^2 (s - c_1)}{2}, \end{aligned}$$
…..……..…………… (2.11)

and hence

$$\int_{b_1}^{b_3} G(u,s)du \leq \frac{(c_2 - c_1)}{2} (c_2 - s)(s - c_1).$$
……………………… (2.12)

Substituting (2.12) in (2.10), we get

$$|y'(t)| \leq \frac{1}{4} \int_{c_1}^{c_2} (c_2 - s)(s - c_1) \left| y^{(4)}(s) \right| ds.$$
…………………… (2.13)

Now, using $a_1 < c_1$ and $c_2 < a_4$ in (2.13), we get

$$|y'(t)| \leq \frac{1}{4} \int_{a_1}^{a_4} (a_4 - s)(s - a_1) \left| y^{(4)}(s) \right| ds.$$
…………………….. (2.14)

Again, integrating (2.14) from a_1 to t and using $\left| \int_{a_1}^{t} y'(u) du \right| \leq \int_{a_1}^{t} |y'(u)|$ du, we get

$$|y(t)| \leq \frac{1}{4} \int_{a_1}^{t} \int_{a_1}^{a_4} (a_4 - s)(s - a_1) \left| y^{(4)}(s) \right| dsdu.$$
…………………… (2.15)

Similarly, we get

$$|y(t)| \leq \frac{1}{4} \int_{t}^{a_4} \int_{a_1}^{a_4} (a_4 - s)(s - a_1) \left| y^{(4)}(s) \right| dsdu.$$
...………………….. (2.16)

Adding (2.15) and (2.16), we get

$$|y(t)| \le \frac{(a_4-a_1)}{8}\int_{a_1}^{a_4}(a_4-s)(s-a_1)\left|y^{(4)}(s)\right|ds. \quad \text{(2.17)}$$

So the proof is completed.

Now, we consider the boundary conditions (1.3). It is clear that from the Rolleís theorem there are $b_4 \in (a_1, a_2)$, and $b_5 \in (a_2, a_3)$ such that

$$y'(a_1) = y'(b_4) = y'(b_5) = 0. \quad \text{(2.18)}$$

Similarly, there are $c_3 \in (a_1, b_4)$ and $c_4 \in (b_4, b_5)$ such that

$$y''(c_1) = y''(c_2) = 0. \quad \text{(2.19)}$$

Lemma 2.2. Assume that $y(t) \in C^4([a_1,a_3],\mathbb{R})$ any function satisfying the boundary conditions (1.3) *(or (1.4), or (1.5)) and* $y(t) \neq 0$ *for* $\forall t \in (a_1,a_2)\cup(a_2,a_3)$. Then the following inequality

$$|y(t)| \le \frac{(a_3-a_1)}{8}\int_{a_1}^{a_3}(a_3-t)(t-a_1)\,|p(t)|\,dt \quad \text{(2.20)}$$

holds.

Proof. The proof of this lemma similar to that of Lemma 2.1. Therefore, it is omitted. □

Main Results

Now, we give the main results.

Theorem 3.1. If y (t) is a nontrivial solution of (1.1) satisfying the four-point boundary conditions (1.2), then the following inequality

$$\frac{8}{(a_4-a_1)} \le \int_{a_1}^{a_4}(a_4-t)(t-a_1)\,|p(t)|\,dt \quad \text{(3.1)}$$

holds.

Proof. Assume that y(t) is a nontrivial solution of (1.1) satisfying the four-point boundary conditions (1.2). From (1.1) and Lemma 2.1, we have

$$\left|y^{(4)}(t)\right| = |p(t)|\,|y(t)| \le |p(t)|\,\frac{(a_4-a_1)}{8}\int_{a_1}^{a_4}(a_4-t)(t-a_1)\left|y^{(4)}(t)\right|dt. \quad \text{(3.2)}$$

Multiplying both sides of (3.2) by $(a_4-t)(t-a_1)$ and integrating from a_1 to a_4, we get

$$\int_{a_1}^{a_4}(a_4-t)(t-a_1)\left|y^{(4)}(t)\right|dt \le \frac{(a_4-a_1)}{8}\int_{a_1}^{a_4}(a_4-t)(t-a_1)\,|p(t)|\,dt\int_{a_1}^{a_4}(a_4-t)(t-a_1)\left|y^{(4)}(t)\right|dt \quad \text{(3.3)}$$

Next, we show that

$$\int_{a_1}^{a_4}(a_4-t)(t-a_1)\left|y^{(4)}(t)\right|dt \neq 0 \quad \text{(3.4)}$$

If (3.4) is not true, then we have

$$\int_{a_1}^{a_4} (a_4 - t)(t - a_1) \left| y^{(4)}(t) \right| dt = 0.$$
………………………… (3.5)

It follows from (2.5) that y(t) $y(t) = 0$ for $\forall t \in [a_1, a_4]$, which is contradicts with the hypotheses since $y(t) \neq 0$ for $\forall t \in (a_1, a_2) \cup (a_2, a_3) \cup (a_3, a_4)$. Hence, by using (3.4) in (3.3), we get the inequality (3.1).

This completes the proof.

Remark 1. It is easy to see that the inequality (3.1) is sharper than the inequalities (1.14). Accordingly, *let* $f(t) = (a_4 - t)(t - a_1)$ *for* $t \in [a_1, a_4]$, and then we take max $\{f(t) : a_1 \leq t \leq a_4\} = \frac{(a_4 - a_1)^2}{4}$ in the inequality (3.1), then we obtain

$$\frac{32}{(a_4 - a_1)^3} \leq \int_{a_1}^{a_4} |p(t)| \, dt.$$
…..……………………….. (3.6)

Let, we give other theorem. Also we note that the proof of this theorem similar to that of Theorem 3.1. Therefore, it is omitted.

Theorem 3.2. If y(t) is a nontrivial solution of (1.1) satisfying the three-point boundary conditions (1.3) (or (1.4), or (1.4)), then the following inequality

$$\frac{8}{(a_3 - a_1)} \leq \int_{a_1}^{a_3} (a_3 - t)(t - a_1) \, |p(t)| \, dt$$
………………………….. (3.7)

holds.

Remark 2. Similarly, we do processes such in Remark 1 for the inequality (3.7), then we get

$$\frac{32}{(a_3 - a_1)^3} \leq \int_{a_1}^{a_3} |p(t)| \, dt.$$
……………………………. (3.8)

The inequality (3.8) is new for nontrivial solution of (1.1) under the boundary conditions (1.3) ((1.4) or (1.5)).

Here, we give an application of the inequalities (3.6) and (3.8) for the following eigenvalue problems

$$y^{(4)}(t) + \lambda p(t) y(t) = 0$$
…………………………. (3.9)

under the fouth-point boundary conditions (1.2) or three-point boundary conditions (1.3) or ((1.4) or (1.5)). Hence, if there exists a nontrivial solution *y*(*t*) of linear homogeneous problems (3.9), then we

have

$$\frac{32}{(a_4 - a_1)^3 \int_{a_1}^{a_4} |p(t)| \, dt.} \leq |\lambda|$$
………………………… (3.10)

and

$$\frac{32}{(a_3 - a_1)^3 \int_{a_1}^{a_3} |p(t)| \, dt.} \leq |\lambda|$$
………………………… (3.11)

respectively.

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