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ON ISOMORPHISM IMPLEMENTED BY MIXED FRACTIONAL INTEGRALS IN HÖLDER SPACES

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ABSTRACT

As is known, the Riemann-Liouville fractional integration operator establishes an isomorphism between Hölder spaces for functions one variables. We study mixed Riemann-Liouville fractional integration operats and mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. The obtained results extend the well known theorem of Hardy-Littlewood for one-dimensuanl fractional integrals to the case of mixed Hölderness.

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INTRODUCTION

The classical result of G.Hardi and D.Littlewood (1928, see [1, §3]) is known that the fractional integral $(I_{a+,0+}^\alpha f)(x) = \Gamma^{-1}(\alpha) (t_+^{\alpha-1} * f)(x)$, $0 < \alpha < 1$ maps isomorphically the space $H_0^\lambda([0, 1])$ of Hölder order $\lambda \in (0, 1)$ functions with a condition $f(0) = 0$ on a similar space of a higher order $\lambda + \alpha$ provided that $\lambda + \alpha < 1$. Further, this result was generalized in various directions: a space with a power weight, generalized Hölder spaces, spaces of the Nikolsky type, etc. A detailed review of these and some other similar results can be found in [1].

In the multidimensional case, the statement about the properties of a map in Hölder spaces for a mixed fractional Riemann – Liouville integral was studied in [2] - [9].

$$(I_{0+,0+}^{\alpha,\beta} \varphi)(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y \frac{\varphi(t, \tau) dt d\tau}{(x-t)^{1-\alpha} (y-\tau)^{1-\beta}}, \quad x > a, y > c, \dots\dots\dots(1)$$

Mixed fractional derivatives form Marchaud ([7]-[9])

$$\frac{\alpha\beta}{\Gamma(1-\alpha)\Gamma(1-\beta)} \int_0^x \int_0^y \frac{\varphi(x, y) - \varphi(t, \tau)}{(x-t)^{1+\alpha} (y-\tau)^{1+\beta}} dt d\tau, \dots\dots\dots(2)$$

where $x > 0, y > 0$, were not studied either in the usual Hölder space, or in the Hölder spaces defined by mixed differences. Meanwhile, there arise "points of interest" related to the investigation of the above mixed differences of fractional derivatives form Marchaud. For operators (1) in Hölder spaces of mixed order there arise some questions to be answered in relation to the

usage of these or those differences in the definition of Hölder spaces. Such mapping properties in Hölder spaces of mixed order were not studied. This paper is aimed to fill in this gap. We deal with non-weighted spaces.

Consider the operator (1) in a rectangle $Q = \{(x, y) : 0 < x < b, 0 < y < d\}$.

For a continuous function $\varphi(x, y)$ on R^2 we introduce the notation

$$\left(\Delta_h \varphi\right)^{(1,0)}(x, y) = \varphi(x + h, y) - \varphi(x, y), \quad \left(\Delta_\eta \varphi\right)^{(0,1)}(x, y) = \varphi(x, y + \eta) - \varphi(x, y),$$

$$\left(\Delta_{h,\eta} \varphi\right)^{(1,1)}(x, y) = \varphi(x + h, y + \eta) - \varphi(x + h, y) - \varphi(x, y + \eta) + \varphi(x, y), \text{ so that}$$

$$\varphi(x + h, y + \eta) = \left(\Delta_{h,\eta} \varphi\right)^{(1,1)}(x, y) + \left(\Delta_h \varphi\right)^{(1,0)}(x, y) + \left(\Delta_\eta \varphi\right)^{(0,1)}(x, y) + \varphi(x, y).$$

.....(3)

Everywhere in the sequel by C, C_1, C_2 etc we denote positive constants which may different values in different occurrences and even in the same line.

Definition 1. Let $\lambda, \gamma \in (0, 1]$. We say that $\varphi \in H^{\lambda, \gamma}(Q)$, if

$$|\varphi(x_1, y_1) - \varphi(x_2, y_2)| \leq C_1|x_1 - x_2|^\lambda + C_2|y_1 - y_2|^\gamma \dots\dots\dots(4)$$

for all $(x_1, y_1), (x_2, y_2) \in Q$. Condition (4) is equivalent to the couple of the separate conditions

$$\left| \left(\Delta_h \varphi\right)^{(1,0)}(x, y) \right| \leq C_1|h|^\lambda, \quad \left| \left(\Delta_\eta \varphi\right)^{(0,1)}(x, y) \right| \leq C_2|\eta|^\gamma$$

uniform with respect to another variable.

By $H_0^{\lambda, \gamma}(Q)$ we define a subspace of functions $f \in H^{\lambda, \gamma}(Q)$, vanishing at the boundaries $x = 0$ and $y = 0$ of Q .

Let $\lambda = 0$ and /or $\gamma = 0$. We put $H^{0,0}(Q) = L^\infty(Q)$ and

$$H^{\lambda, 0}(Q) = \left\{ \varphi \in L^\infty(Q) : \left| \left(\Delta_h \varphi\right)^{(1,0)}(x, y) \right| \leq C_1|h|^\lambda \right\}, \quad \lambda \in (0, 1],$$

$$H^{0, \gamma}(Q) = \left\{ \varphi \in L^\infty(Q) : \left| \left(\Delta_\eta \varphi\right)^{(0,1)}(x, y) \right| \leq C_2|\eta|^\gamma \right\}, \quad \gamma \in (0, 1].$$

Definition 2. We say that $\varphi(x, y) \in \tilde{H}^{\lambda, \gamma}(Q)$, where $\lambda, \gamma \in (0, 1]$, if

$$\varphi(x, y) \in H^{\lambda, \gamma}(Q) \text{ and } \left| \left(\Delta_{h,\eta} \varphi\right)^{(1,1)}(x, y) \right| \leq C_3|h|^\lambda|\eta|^\gamma. \dots\dots\dots(5)$$

We say that $\varphi(x, y) \in \tilde{H}_0^{\lambda, \gamma}(Q)$, if $\varphi(x, y) \in \tilde{H}^{\lambda, \gamma}(Q)$ and $\varphi(x, y)|_{x=0, y=0} = 0$.

These spaces become Banach spaces under the standard definition of the

$$\|\varphi\|_{H^{\lambda, \gamma}} := \|\varphi\|_{C(Q)} + \sup_{x, x+h \in [0, b]} \sup_{y \in [0, d]} \frac{\left| \left(\Delta_h \varphi\right)^{(1,0)}(x, y) \right|}{|h|^\lambda} + \sup_{x \in [0, b]} \sup_{y, y+\eta \in [0, d]} \frac{\left| \left(\Delta_\eta \varphi\right)^{(0,1)}(x, y) \right|}{|\eta|^\gamma},$$

$$\|\varphi\|_{\tilde{H}^{\lambda,\gamma}} := \|\varphi\|_{H^{\lambda,\gamma}} + \sup_{x, x+h \in [0, b]} \sup_{y, y+\eta \in [0, d]} \frac{\left| \left(\Delta_{h,\eta}^{1,1} \varphi \right) (x, y) \right|}{|h|^\lambda |\eta|^\gamma}.$$

Note that

$$\varphi(x, y) \in \tilde{H}^{\lambda,\gamma} \Rightarrow \left| \left(\Delta_{h,\eta}^{1,1} \varphi \right) (x, y) \right| \leq C_\theta |h|^{\theta\lambda} |\eta|^{(1-\theta)\gamma}, \text{ for any } \theta \in [0, 1], \tag{6}$$

where $C_\theta = 2C_1^\theta C_2^{1-\theta}$, so that

$$\bigcap_{0 \leq \theta \leq 1} \tilde{H}^{\theta\lambda, (1-\theta)\gamma}(\mathcal{Q}) \lrcorner H^{\lambda,\gamma}(\mathcal{Q}) \lrcorner \tilde{H}^{\lambda,\gamma}(\mathcal{Q}) \tag{7}$$

where \lrcorner stands for the continuous embedding, and the norm for $\bigcap_{0 \leq \theta \leq 1} \tilde{H}^{\theta\lambda, (1-\theta)\gamma}(\mathcal{Q})$ is introduced as the maximum in θ of norms

for $\tilde{H}^{\theta\lambda, (1-\theta)\gamma}(\mathcal{Q})$. Since $\theta \in [0, 1]$ is arbitrary, it is not hard to see that the inequality in (6) is equivalent to

$$\left| \left(\Delta_{h,\eta}^{1,1} \varphi \right) (x, y) \right| \leq C \min \{ |h|^\lambda, |\eta|^\gamma \}. \tag{8}$$

Mapping properties of the mixed fractional integration operator in Hölder spaces.

Theorem 1. Let $\varphi(x, y) \in H^{\lambda,\gamma}(\mathcal{Q})$, $0 \leq \lambda, \gamma \leq 1$, $0 < \alpha, \beta < 1$. Then for the mixed fractional integral operator (1) the representation

$$\left(I_{0+,0+}^{\alpha,\beta} \varphi \right) (x, y) = \frac{\varphi(0,0)}{\Gamma(1+\alpha)\Gamma(1+\beta)} x^\alpha y^\beta + \frac{y^\beta}{\Gamma(1+\beta)} \psi_1(x) + \frac{x^\alpha}{\Gamma(1+\alpha)} \psi_2(y) + \psi(x, y) \tag{9}$$

holds, where

$$\psi_1(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t, \cdot) - \varphi(0,0)}{(x-t)^{1-\alpha}} dt, \quad \psi_2(y) = \frac{1}{\Gamma(\beta)} \int_c^y \frac{\varphi(0,s) - \varphi(0,0c)}{(y-s)^{1-\beta}} ds,$$

$$\psi(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y \frac{\left(\Delta_{t,s}^{1,1} \varphi \right) (0,0)}{(x-t)^{1-\alpha} (y-s)^{1-\beta}} dt ds,$$

and

$$|\psi_1(x)| \leq C_1 x^{\lambda+\alpha}, \quad |\psi_2(y)| \leq C_2 y^{\gamma+\beta}, \tag{10}$$

$$|\psi(x, y)| \leq C \min_{0 \leq \theta \leq 1} x^{\alpha+\theta\lambda} y^{\beta+(1-\theta)\gamma} = C x^\alpha y^\beta \min \{ x^\lambda, y^\gamma \}. \tag{11}$$

Proof. Representation (9) itself is easily obtained by means of (3). Since $\varphi \in H^{\lambda,\gamma}(\mathcal{Q})$, inequalities (10) are obvious. Estimate (11) is obtained by means of (6) and (8).

Theorem 2. Let $0 \leq \lambda, \gamma \leq 1$. Then the mixed fractional integral operator $I_{0+,0+}^{\alpha,\beta}$ is bounded from $H_0^{\lambda,\gamma}(\mathcal{Q})$ into $H_0^{\lambda+\alpha,\gamma+\beta}(\mathcal{Q})$, if $\lambda + \alpha < 1$ and $\gamma + \beta < 1$.

Proof. Since $\varphi(x, y) \in H_0^{\lambda,\gamma}(\mathcal{Q})$, then by (9) we have

$$\left(I_{a+,c+}^{\alpha,\beta} \varphi \right) (x, y) = \psi(x, y).$$

We denote

$$g(x, y) = \left(\Delta_{x,y}^{1,1} \varphi \right) (0,0) \tag{12}$$

for brevity. Note that

$$\left(\Delta_{x,y}^{1,1} \varphi\right)(0,0) = \varphi(x,y)$$

for $\varphi \in H_0^{\lambda,\gamma}$, but we prefer to keep the notation for $g(x,y)$ via the mixed difference as in (12). By (6) we have

$$|g(x,y)| \leq Cx^{\theta\lambda}y^{(1-\theta)\gamma} \leq \min\{x^\lambda, y^\gamma\}. \tag{13}$$

For $h > 0$; $x, x+h \in Q_1 = [0, b]$, we consider the difference

$$\begin{aligned} \psi(x+h,y) - \psi(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left(\int_{-h}^x \int_0^y \frac{g(x-t,y-s)}{(t+h)^{1-\alpha}s^{1-\beta}} dt ds - \right. \\ &\quad \left. - \int_0^x \int_0^y \frac{g(x-t,y-s)}{t^{1-\alpha}s^{1-\beta}} dt ds \right) = \frac{(x+h)^\alpha - x^\alpha}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^y \frac{g(x,y-s)}{s^{1-\beta}} ds + \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{-h}^0 \int_0^y \frac{g(x-t,y-s) - g(x,y-s)}{(t+h)^{1-\alpha}s^{1-\beta}} dt ds + \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y \frac{g(x-t,y-s) - g(x,y-s)}{s^{1-\beta}} [(t+h)^{\alpha-1} - t^{\alpha-1}] dt ds = \Delta_1 + \Delta_2 + \Delta_3. \end{aligned} \tag{14}$$

Since $\forall \theta \in [0, 1]$, we make use of (13) with $\theta=1$ and obtain

$$|\Delta_1| \leq Cx^\lambda |(x+h)^\alpha - x^\alpha| \leq Ch^{\alpha+\lambda}.$$

For Δ_2 in view of (6), we have

$$|g(x-t,y-s) - g(x,y-s)| = \left| \left(\Delta_{-t,y-s}^{1,1} \varphi\right)(x,0) \right| \leq C|t|^\lambda, \tag{15}$$

and then $\Delta_2 \leq Ch^{\lambda+\alpha}$.

For Δ_3 by (15) and (6) we obtain

$$\Delta_3 \leq C \int_0^x t^\lambda |t^{\alpha-1} - (t+h)^{\alpha-1}| dt \leq C_0 h^{\alpha+\lambda}, \quad C_0 = \int_0^\infty t^\lambda |t^{\alpha-1} - (t+h)^{\alpha-1}| dt < \infty.$$

Gathering the estimates $\Delta_1, \Delta_2, \Delta_3$ we obtain

$$|\psi(x+h,y) - \psi(x,y)| \leq C_1 h^{\lambda+\alpha}.$$

Rearranging symmetrically representation (14), we can similarly obtain that

$$|\psi(x,y+\eta) - \psi(x,y)| \leq C_2 \eta^{\beta+\gamma}.$$

Which proves the theorem.

Theorem 3. The mixed fractional integral operator $I_{0+,0+}^{\alpha,\beta}$ is bounded from the sapce $\tilde{H}_0^{\lambda,\gamma}(Q)$, $0 \leq \lambda, \gamma \leq 1$ into the space $\tilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$, if $\lambda + \alpha < 1$ and $\gamma + \beta < 1$.

Proof. Let $\varphi(x, y) \in \tilde{H}_0^{\lambda, \gamma}(Q)$. By Theorem 2 and embedding (7), for $f(x, y) = (I_{0+, 0+}^{\alpha, \beta} \varphi)(x, y)$ it satisfies to estimate the difference $\left(\Delta_{h, \eta}^{1,1} f \right)(x, y)$. Since $\varphi(x, y)|_{x=0, y=0} = 0$, according to (9) we have $f(x, y) = \psi(x, y)$, where $\psi(x, y)$ is the function from (9). the main moment in the estimations is to find the corresponding splitting which allows to derive the best information in each variable not losing the corresponding information in another variable. The suggested splitting runs as follows

$$\begin{aligned} \left(\Delta_{h, \eta}^{1,1} f \right)(x, y) &= \left(\Delta_{h, \eta}^{1,1} \psi \right)(x, y) = \sum_{k=1}^9 T_k := \\ &= \frac{g(x, y) \left[(x+h)^\alpha - x^\alpha \right] \left[(y+\eta)^\beta - y^\beta \right]}{\Gamma(1+\alpha)\Gamma(1+\beta)} + \frac{(y+\eta)^\beta - y^\beta}{\Gamma(\alpha)\Gamma(1+\beta)} \int_{-h}^0 \frac{g(x-t, y) - g(x, y)}{(t+h)^{1-\alpha}} dt + \\ &+ \frac{(x+h)^\alpha - x^\alpha}{\Gamma(\alpha+1)\Gamma(\beta)} \int_{-\eta}^0 \frac{g(x, y-s) - g(x, y)}{(s+\eta)^{1-\beta}} ds + \\ &+ \frac{(y+\eta)^\beta - y^\beta}{\Gamma(\alpha)\Gamma(\beta+1)} \int_0^x [g(x-t, y) - g(x, y)] [(t+h)^{\alpha-1} - t^{\alpha-1}] dt + \\ &+ \frac{(x+h)^\alpha - x^\alpha}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^y [g(x, y-s) - g(x, y)] [(s+\eta)^{\beta-1} - s^{\beta-1}] ds + \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{-h}^0 \int_{-\eta}^0 \frac{\left(\Delta_{-t, -s}^{1,1} g \right)(x, y)}{(h+t)^{1-\alpha} (s+\eta)^{1-\beta}} dt ds + \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{-h}^0 \int_0^y \frac{\left(\Delta_{-t, -s}^{1,1} g \right)(x, y)}{(h+t)^{1-\alpha}} [(s+\eta)^{\beta-1} - s^{\beta-1}] dt ds + \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_{-\eta}^0 \frac{\left(\Delta_{-t, -s}^{1,1} g \right)(x, y)}{(s+\eta)^{1-\beta}} [(t+h)^{\alpha-1} - t^{\alpha-1}] dt ds + \\ &+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y \left(\Delta_{-t, -s}^{1,1} g \right)(x, y) [(h+t)^{\alpha-1} - t^{\alpha-1}] [(s+\eta)^{\beta-1} - s^{\beta-1}] dt ds, \end{aligned}$$

where $h, \eta > 0$; $x, x+h \in [0, b]$, $y, y+\eta \in [0, d]$ and $g(x, y)$ is the function from (12). The validity of this representation may be checked directly.

Since $\varphi(x, y) \in \tilde{H}_0^{\lambda, \gamma}$, we have

$$|g(x, y)| = \left| \left(\Delta_{x, y}^{1,1} \varphi \right)(0, 0) \right| \leq Cx^\lambda y^\gamma \dots\dots\dots(16)$$

and then

$$\begin{aligned} |T_1| &\leq Cx^\lambda [(x+h)^\alpha - x^\alpha] y^\gamma [(y+\eta)^\beta - y^\beta], \\ |T_2| &\leq Cy^\gamma [(y+\eta)^\beta - y^\beta] \int_{-h}^0 \frac{|t|^\lambda}{(t+h)^{1-\alpha}} dt, \\ |T_3| &\leq Cx^\lambda [(x+h)^\alpha - x^\alpha] \int_{-\eta}^0 \frac{|s|^\gamma}{(s+\eta)^{1-\beta}} ds, \\ |T_4| &\leq Cy^\gamma [(y+\eta)^\beta - y^\beta] \int_0^x t^\lambda |(t+h)^{\alpha-1} - t^{\alpha-1}| dt, \end{aligned}$$

$$|T_5| \leq Cx^\lambda \left[(x+h)^\alpha - x^\alpha \right] \int_0^y s^\gamma |(s+\eta)^{\beta-1} - s^{\beta-1}| ds .$$

For $T_6 - T_9$ we similarly, make use of

$$\left| \left(\Delta_{-t, -s}^{1,1} g \right) (x, y) \right| = \left| \left(\Delta_{-t, -s}^{1,1} \phi \right) (x, y) \right| \leq C |t|^\lambda |s|^\gamma , \tag{17}$$

and obtain

$$|T_6| \leq C \int_{-h}^0 \frac{|t|^\lambda}{(h+t)^{1-\alpha}} dt \int_{-\eta}^0 \frac{|s|^\gamma}{(s+\eta)^{1-\beta}} ds ,$$

$$|T_7| \leq C \int_{-h}^0 \frac{|t|^\lambda}{(h+t)^{1-\alpha}} dt \int_0^y s^\gamma |(s+\eta)^{\beta-1} - s^{\beta-1}| ds ,$$

$$|T_8| \leq C \int_0^x t^\lambda |(t+h)^{\alpha-1} - t^{\alpha-1}| dt \int_{-\eta}^0 \frac{|s|^\gamma}{(s+\eta)^{1-\beta}} ds ,$$

$$|T_9| \leq C \int_0^x t^\lambda |(h+t)^{\alpha-1} - t^{\alpha-1}| dt \int_0^y s^\gamma |(s+\eta)^{\beta-1} - s^{\beta-1}| ds ,$$

after which every term is estimated in the standard way, and we get

$$\left| \left(\Delta_{h, \eta}^{1,1} f \right) (x, y) \right| \leq C_3 h^{\lambda+\alpha} \eta^{\gamma+\beta} . \tag{18}$$

This completes the proof.

Mapping properties of the mixed fractional differentiation operator in Hölder spaces.

Theorem 4. Let $f(x, y) \in \tilde{H}^{\lambda, \gamma}(Q)$, $\alpha < \lambda \leq 1, \beta < \gamma \leq 1$. Then for the mixed fractional differential operator (2) the representation

$$\left(D_{0+, 0+}^{\alpha, \beta} f \right) (x, y) = \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \left[\frac{f(0,0)}{x^\alpha y^\beta} + \frac{\psi_1(x)}{y^\beta} + \frac{\psi_2(y)}{x^\alpha} + \psi(x, y) \right] , \tag{19}$$

and

$$|\psi_1(x)| \leq C_1 x^{\lambda-\alpha}, \quad |\psi_2(y)| \leq C_2 y^{\gamma-\beta}, \tag{20}$$

$$|\psi(x, y)| \leq C x^{\lambda-\alpha} y^{\gamma-\beta} \tag{21}$$

where

$$\psi_1(x) = \frac{f(x,0) - f(0,0)}{x^\alpha} + \alpha \int_0^x \frac{f(x,0) - f(t,0)}{(x-t)^{\alpha+1}} dt ,$$

$$\psi_2(y) = \frac{f(0,y) - f(0,0)}{y^\beta} + \beta \int_0^y \frac{f(0,y) - f(0,\tau)}{(y-\tau)^{1+\beta}} d\tau ,$$

$$\psi(x, y) = \frac{1}{x^\alpha y^\beta} \left(\Delta_{x,y}^{1,1} f \right) (0,0) + \frac{\alpha}{y^\beta} \int_0^x \left(\Delta_{x-t,y}^{1,1} f \right) (t,0) \frac{dt}{(x-t)^{1+\alpha}} +$$

$$+ \frac{\beta}{x^\alpha} \int_0^y \left(\Delta_{x,y-\tau}^{1,1} f \right) (0,\tau) \frac{d\tau}{(y-\tau)^{1+\beta}} + \alpha\beta \int_0^x \int_0^y \left(\Delta_{x-t,y-\tau}^{1,1} f \right) (t,\tau) \frac{dtd\tau}{(x-t)^{1+\alpha} (y-\tau)^{1+\beta}} .$$

Proof. Representation (19) itself is easily obtained by means of (3). Since $f \in H^{\lambda, \gamma}(Q)$, inequalities (2) are obvious. Estimate (21) is obtained by means of (6) and (8).

Theorem 5. Let $f(x, y) \in H_0^{\lambda+\alpha, \gamma+\beta}(Q)$, $\alpha < \lambda \leq 1$, $\beta < \gamma \leq 1$. Then the operator $D_{0+,0+}^{\alpha, \beta}$ continuously maps $H_0^{\lambda+\alpha, \gamma+\beta}(Q)$ into $H_0^{\lambda, \gamma}(Q)$.

Proof. Since $f(x, y) \in H_0^{\lambda+\alpha, \gamma+\beta}(Q)$, by (19) we have

$$\varphi(x, y) = (D_{0+,0+}^{\alpha, \beta} f)(x, y) = \psi(x, y).$$

By (6) and (8) we have

$$\left| \left(\Delta_{x,y}^{1,1} f \right) (0,0) \right| \leq C x^{\theta(\lambda+\alpha)} y^{(1-\theta)(\gamma+\beta)} \leq \min \{ x^{\lambda+\alpha}, y^{\gamma+\beta} \}. \tag{22}$$

Let $h > 0$; $x, x+h \in [0, b]$. We consider the difference

$$\begin{aligned} \psi(x+h, y) - \psi(x, y) &= \frac{\left(\Delta_{h,y}^{1,1} f \right) (0,0)}{y^\beta (x+h)^\alpha} + \frac{\left(\Delta_{x,y}^{1,1} f \right) (0,0)}{y^\beta} \left[\frac{1}{(x+h)^\alpha} - \frac{1}{x^\alpha} \right] + \\ &+ \frac{\alpha}{y^\beta} \int_0^x \frac{\left(\Delta_{h,y}^{1,1} f \right) (x,0)}{(x+h-t)^{1+\alpha}} dt + \frac{\alpha}{y^\beta} \int_x^{x+h} \frac{\left(\Delta_{x+h-t,y}^{1,1} f \right) (t,0)}{(x+h-t)^{1+\alpha}} dt + \frac{\beta}{(x+h)^\alpha} \int_0^y \frac{\left(\Delta_{h,y-\tau}^{1,1} f \right) (0,\tau)}{(y-\tau)^{1+\beta}} d\tau + \\ &+ \frac{\alpha}{y^\beta} \int_0^x \left(\Delta_{x-t,y}^{1,1} f \right) (t,0) \left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha} \right] dt + \beta \left[(x+h)^\alpha - x^\alpha \right] \int_0^y \frac{\left(\Delta_{x,y-\tau}^{1,1} f \right) (0,\tau)}{(y-\tau)^{1+\beta}} d\tau + \\ &+ \alpha \beta \int_0^x \int_0^y \frac{\left(\Delta_{h,y-\tau}^{1,1} f \right) (x,\tau) dt d\tau}{(x+h-t)^{1+\alpha} (y-\tau)^{1+\beta}} + \alpha \beta \int_x^{x+h} \int_0^y \frac{\left(\Delta_{x+h-t,y-\tau}^{1,1} f \right) (x,\tau) dt d\tau}{(x+h-t)^{1+\alpha} (y-\tau)^{1+\beta}} + \\ &+ \alpha \beta \int_0^x \int_0^y \frac{\left(\Delta_{x-t,y-\tau}^{1,1} f \right) (x,\tau)}{(y-\tau)^{1+\beta}} \left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha} \right] dt d\tau. \tag{23} \end{aligned}$$

We make use of (22) with $\theta = 1$ and obtain

$$\begin{aligned} |\psi(x+h, y) - \psi(x, y)| &\leq C_1 \left[\frac{h^{\lambda+\alpha}}{(x+h)^\alpha} + x^{\lambda+\alpha} \left[\frac{1}{(x+h)^\alpha} - \frac{1}{x^\alpha} \right] + h^{\lambda+\alpha} \int_0^x \frac{dt}{(x+h-t)^{1+\alpha}} + \right. \\ &\left. + \int_x^{x+h} \frac{dt}{(x+h-t)^{1-\lambda}} + \int_0^x (x-t)^{\lambda+\alpha} \left[\frac{1}{(x+h-t)^{1+\alpha}} - \frac{1}{(x-t)^{1+\alpha}} \right] dt \right] = \sum_{k=1}^5 \Phi_k. \end{aligned}$$

For Φ_1 we have

$$\Phi_1 = C_1 h^\lambda \left(\frac{h}{x+h} \right)^\alpha \leq Ch^\lambda$$

Let's estimate Φ_2 . Here we shall consider two case: $x \leq h$ and $x > h$. In the first case, we use inequality

$$|\sigma_1^\mu - \sigma_2^\mu| \leq |\sigma_1 - \sigma_2|^\mu, \quad (\sigma_1 \neq \sigma_2)$$

and obtain

$$\Phi_2 \leq C_1 \frac{x^\lambda h^\alpha}{(x+h)^\alpha} \leq Ch^\lambda,$$

in second case, using $(1+t)^\alpha - 1 \leq \alpha t$, $t > 0$ we have

$$\Phi_2 \leq C_1 \frac{x^{\lambda+\alpha-1} h}{(x+h)^\alpha} \leq Ch^\lambda.$$

For Φ_3 we have:

$$\Phi_3 \leq C_1 h^{\lambda+\alpha} \int_0^x \frac{dt}{(t+h)^{1+\alpha}} \leq Ch^\lambda.$$

For Φ_4

$$\Phi_4 = C_1 \int_x^{x+h} (x+h-t)^{\lambda-1} dt = Ch^\lambda.$$

For Φ_5

$$\Phi_5 = C_1 h^\lambda \int_0^{x/h} t^\lambda \left| \frac{1}{(1+t)^{1+\alpha}} - \frac{1}{t^{1+\alpha}} \right| dt \leq Ch^\lambda.$$

Gathering the estimates for $\Phi_k, k = \overline{1,5}$ we obtain $|\psi(x+h, y) - \psi(x, y)| \leq Ch^\lambda$.

Rearranging symmetrically representation (12), we can similarly obtain that $|\psi(x, y+\eta) - \psi(x, y)| \leq C\eta^\gamma$.

Theorem 6. Let $f(x, y) \in \tilde{H}_0^{\lambda+\alpha, \gamma+\beta}(Q), \alpha < \lambda \leq 1, \beta < \gamma \leq 1$. Then the operator $\mathbf{D}_{0+,0+}^{\alpha,\beta}$ continuously maps $\tilde{H}_0^{\lambda+\alpha, \gamma+\beta}(Q)$ into $\tilde{H}_0^{\lambda, \gamma}(Q)$.

Proof. Let $f(x, y) \in \tilde{H}_0^{\lambda+\alpha, \gamma+\beta}(Q)$. Then we have $\Phi(x, y) = (\mathbf{D}_{0+,0+}^{\alpha,\beta} f)(x, y) = \psi(x, y)$, where $\psi(x, y)$ is the function from (9). The main moment in the estimations is to find the corresponding splitting which allows to derive the best information in each variable not losing the corresponding information in another variable.

Let $h, \eta > 0; x, x+h \in [0, b], y, y+\eta \in [0, d]$. We consider the difference

$$\begin{aligned} \left(\Delta_{h,\eta} \psi \right)(x, y) &= \sum_{k=1}^{25} \Psi_k := \frac{\left(\Delta_{h,\eta} f \right)(x, y)}{(x+h)^\alpha (y+\eta)^\beta} + \frac{\left(\Delta_{h,y} f \right)(x, 0)}{(x+h)^\alpha} \left[\frac{1}{y^\beta} - \frac{1}{(y+\eta)^\beta} \right] + \\ &+ \frac{\left(\Delta_{x,\eta} f \right)(0, y)}{(y+\eta)^\beta} \left[\frac{1}{x^\alpha} - \frac{1}{(x+h)^\alpha} \right] + \left(\Delta_{x,y} f \right)(0, 0) \left[\frac{1}{x^\alpha} - \frac{1}{(x+h)^\alpha} \right] \left[\frac{1}{y^\beta} - \frac{1}{(y+\eta)^\beta} \right] + \\ &+ \frac{\beta}{(x+h)^\alpha} \int_y^{y+\eta} \frac{\left(\Delta_{h, y+\eta-\tau} f \right)(x, \tau)}{(y+\eta-\tau)^{1+\beta}} d\tau + \frac{\beta}{(x+h)^\alpha} \int_0^y \frac{\left(\Delta_{h,\eta} f \right)(x, y)}{(y+\eta-\tau)^{1+\beta}} d\tau + \\ &+ \beta \left[\frac{1}{x^\alpha} - \frac{1}{(x+h)^\alpha} \right] \int_y^{y+\eta} \frac{\left(\Delta_{x, y+\eta-\tau} f \right)(0, \tau)}{(y+\eta-\tau)^{1+\beta}} d\tau + \\ &+ \frac{\beta}{(x+h)^\alpha} \int_0^y \frac{\left(\Delta_{h, y-\tau} f \right)(x, \tau)}{(y-\tau)^{1+\beta}} \left[\frac{1}{(y-\tau)^{1+\beta}} - \frac{1}{(y+\eta-\tau)^{1+\beta}} \right] d\tau + \\ &+ \beta \left[\frac{1}{x^\alpha} - \frac{1}{(x+h)^\alpha} \right] \int_0^y \frac{\left(\Delta_{x,\eta} f \right)(0, y)}{(y+\eta-\tau)^{1+\beta}} d\tau + \frac{\alpha}{(y+\eta)^\beta} \int_x^{x+h} \frac{\left(\Delta_{x+h-t,\eta} f \right)(t, y)}{(x+h-t)^{1+\alpha}} dt + \\ &+ \beta \left[\frac{1}{x^\alpha} - \frac{1}{(x+h)^\alpha} \right] \int_0^y \frac{\left(\Delta_{x, y-\tau} f \right)(0, \tau)}{(y+\eta)^{1+\beta}} \left[\frac{1}{y^{1+\beta}} - \frac{1}{(y+\eta)^{1+\beta}} \right] d\tau + \end{aligned}$$

$$\begin{aligned}
 & + \frac{\alpha}{(y + \eta)^\beta} \int_0^x \frac{\left(\Delta_{h, \eta}^{1,1} f \right) (x, y)}{(x + h - t)^{1+\alpha}} dt + \alpha \left[\frac{1}{y^\beta} - \frac{1}{(y + \eta)^\beta} \right] \int_x^{x+h} \frac{\left(\Delta_{x+h-t, y}^{1,1} f \right) (t, 0)}{(x + h - t)^{1+\alpha}} dt + \\
 & + \frac{\alpha}{(y + \eta)^\beta} \int_0^x \left(\Delta_{x-t, \eta}^{1,1} f \right) (t, y) \left[\frac{1}{(x - t)^{1+\alpha}} - \frac{1}{(x + h - t)^{1+\alpha}} \right] dt + \\
 & + \alpha \left[\frac{1}{y^\beta} - \frac{1}{(y + \eta)^\beta} \right] \int_0^x \frac{\left(\Delta_{h, y}^{1,1} f \right) (x, 0)}{(x + h - t)^{1+\alpha}} dt + \\
 & + \alpha \left[\frac{1}{y^\beta} - \frac{1}{(y + \eta)^\beta} \right] \int_0^x \left(\Delta_{x-t, y}^{1,1} f \right) (t, 0) \left[\frac{1}{(x - t)^{1+\alpha}} - \frac{1}{(x + h - t)^{1+\alpha}} \right] dt + \\
 & + \int_0^x \int_0^y \frac{\left(\Delta_{h, \eta}^{1,1} f \right) (x, y) dt d\tau}{(x + h - t)^{1+\alpha} (y + \eta - \tau)^{1+\beta}} + \int_0^x \int_y^{y+\eta} \frac{\left(\Delta_{h, y+\eta-\tau}^{1,1} f \right) (x, \tau) dt d\tau}{(x + h - t)^{1+\alpha} (y + \eta - \tau)^{1+\beta}} + \\
 & + \int_0^x \int_0^y \frac{\left(\Delta_{h, y-\tau}^{1,1} f \right) (x, \tau) \left[\frac{1}{(y - \tau)^{1+\beta}} - \frac{1}{(y + \eta - \tau)^{1+\beta}} \right] dt d\tau}{(x + h - t)^{1+\alpha}} + \\
 & + \int_x^{x+h} \int_0^y \frac{\left(\Delta_{x+h-t, \eta}^{1,1} f \right) (t, y) dt d\tau}{(x + h - t)^{1+\alpha} (y + \eta - \tau)^{1+\beta}} + \int_x^{x+h} \int_y^{y+\eta} \frac{\left(\Delta_{x+h-t, y+\eta-\tau}^{1,1} f \right) (t, \tau) dt d\tau}{(x + h - t)^{1+\alpha} (y + \eta - \tau)^{1+\beta}} + \\
 & + \int_x^{x+h} \int_0^y \frac{\left(\Delta_{x+h-t, y-\tau}^{1,1} f \right) (t, \tau) \left[\frac{1}{(y - \tau)^{1+\beta}} - \frac{1}{(y + \eta - \tau)^{1+\beta}} \right] dt d\tau}{(x + h - t)^{1+\alpha}} + \\
 & + \int_0^x \int_0^y \frac{\left(\Delta_{x-t, \eta}^{1,1} f \right) (t, y) \left[\frac{1}{(y + \eta - \tau)^{1+\beta}} \left[\frac{1}{(x - t)^{1+\alpha}} - \frac{1}{(x + h - t)^{1+\alpha}} \right] \right] dt d\tau}{(y + \eta - \tau)^{1+\beta}} + \\
 & + \int_0^x \int_y^{y+\eta} \frac{\left(\Delta_{x-t, y+\eta-\tau}^{1,1} f \right) (t, \tau) \left[\frac{1}{(y + \eta - \tau)^{1+\beta}} \left[\frac{1}{(x - t)^{1+\alpha}} - \frac{1}{(x + h - t)^{1+\alpha}} \right] \right] dt d\tau}{(y + \eta - \tau)^{1+\beta}} + \\
 & + \int_0^x \int_0^y \left(\Delta_{x-t, y-\tau}^{1,1} f \right) (t, \tau) \left[\frac{1}{(x - t)^{1+\alpha}} - \frac{1}{(x + h - t)^{1+\alpha}} \right] \left[\frac{1}{(y - \tau)^{1+\beta}} - \frac{1}{(y + \eta - \tau)^{1+\beta}} \right] dt d\tau .
 \end{aligned}$$

The validity of this representation may be checked directly. Since $f(x, y) \in \tilde{H}_0^{\lambda, \gamma}(Q)$, we have

$$\begin{aligned}
 \left| \left(\Delta_{h, \eta}^{1,1} \psi \right) (x, y) \right| & \leq \sum_{k=1}^{25} |\Psi_k| \leq C \left[\frac{h^\lambda \eta^\gamma}{(x + h)^\alpha (y + \eta)^\beta} + \frac{h^\lambda y^\gamma}{(x + h)^\alpha} \left[\frac{1}{y^\beta} - \frac{1}{(y + \eta)^\beta} \right] + \right. \\
 & + \left. \frac{x^\lambda \eta^\gamma}{(y + \eta)^\beta} \left[\frac{1}{x^\alpha} - \frac{1}{(x + h)^\alpha} \right] + x^\lambda y^\gamma \left[\frac{1}{x^\alpha} - \frac{1}{(x + h)^\alpha} \right] \left[\frac{1}{y^\beta} - \frac{1}{(y + \eta)^\beta} \right] + \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{h^\lambda}{(x+h)^\alpha} \int_y^{y+\eta} (y+\eta-\tau)^{\gamma-1-\beta} d\tau + \frac{\beta h^\lambda \eta^\gamma}{(x+h)^\alpha} \int_0^y \frac{d\tau}{(y+\eta-\tau)^{1+\beta}} + \beta x^\lambda \left[\frac{1}{x^\alpha} - \frac{1}{(x+h)^\alpha} \right] \int_y^{y+\eta} (y+\eta-\tau)^{\gamma-1-\beta} d\tau + \\
& + \frac{h^\lambda \beta}{(x+h)^\alpha} \int_0^y (y-\tau)^\gamma \left[\frac{1}{(y-\tau)^{1+\beta}} - \frac{1}{(y+\eta-\tau)^{1+\beta}} \right] d\tau + \\
& + x^\lambda \eta^\gamma \beta \left[\frac{1}{x^\alpha} - \frac{1}{(x+h)^\alpha} \right] \int_0^y \frac{d\tau}{(y+\eta-\tau)^{1+\beta}} + \frac{\eta^\gamma \alpha}{(y+\eta)^\beta} \int_x^{x+h} (x+h-t)^{\lambda-1-\alpha} dt + \\
& + x^\lambda \beta \left[\frac{1}{x^\alpha} - \frac{1}{(x+h)^\alpha} \right] \int_0^y (y-\tau)^\gamma \left[\frac{1}{y^{1+\beta}} - \frac{1}{(y+\eta)^{1+\beta}} \right] d\tau + \\
& + \frac{\alpha h^\lambda \eta^\gamma}{(y+\eta)^\beta} \int_0^x \frac{dt}{(x+h-t)^{1+\alpha}} + \alpha y^\gamma \left[\frac{1}{y^\beta} - \frac{1}{(y+\eta)^\beta} \right] \int_x^{x+h} (x+h-t)^{\lambda-1-\alpha} dt + \\
& + \frac{\alpha \eta^\gamma}{(y+\eta)^\beta} \int_0^x (x-t)^\lambda \left[\frac{1}{(x-t)^{1+\alpha}} - \frac{1}{(x+h-t)^{1+\alpha}} \right] dt + \\
& + \alpha h^\lambda y^\gamma \left[\frac{1}{y^\beta} - \frac{1}{(y+\eta)^\beta} \right] \int_0^x \frac{dt}{(x+h-t)^{1+\alpha}} + \\
& + \alpha y^\gamma \left[\frac{1}{y^\beta} - \frac{1}{(y+\eta)^\beta} \right] \int_0^x (x-t)^\lambda \left[\frac{1}{(x-t)^{1+\alpha}} - \frac{1}{(x+h-t)^{1+\alpha}} \right] dt + \\
& + h^\lambda \eta^\gamma \int_0^x \int_0^y \frac{dtd\tau}{(x+h-t)^{1+\alpha} (y+\eta-\tau)^{1+\beta}} + \int_0^x \int_y^{y+\eta} \frac{h^\lambda dtd\tau}{(x+h-t)^{1+\alpha} (y+\eta-\tau)^{1+\beta-\gamma}} + \\
& + \int_0^x \int_0^y \frac{h^\lambda (y-\tau)^\gamma}{(x+h-t)^{1+\alpha}} \left[\frac{1}{(y-\tau)^{1+\beta}} - \frac{1}{(y+\eta-\tau)^{1+\beta}} \right] dtd\tau + \\
& + \int_x^{x+h} \int_0^y \frac{\left(\Delta_{x+h-t, \eta}^{1,1} f \right) (t, y) dtd\tau}{(x+h-t)^{1+\alpha} (y+\eta-\tau)^{1+\beta}} + \int_x^{x+h} \int_y^{y+\eta} \frac{\left(\Delta_{x+h-t, y+\eta-\tau}^{1,1} f \right) (t, \tau) dtd\tau}{(x+h-t)^{1+\alpha} (y+\eta-\tau)^{1+\beta}} + \\
& + \int_x^{x+h} \int_0^y \frac{(y-\tau)^\gamma}{(x+h-t)^{1+\alpha-\lambda}} \left[\frac{1}{(y-\tau)^{1+\beta}} - \frac{1}{(y+\eta-\tau)^{1+\beta}} \right] dtd\tau + \\
& + \int_0^x \int_0^y \frac{(x-t)^\lambda \eta^\gamma}{(y+\eta-\tau)^{1+\beta}} \left[\frac{1}{(x-t)^{1+\alpha}} - \frac{1}{(x+h-t)^{1+\alpha}} \right] dtd\tau + \\
& + \int_0^x \int_y^{y+\eta} \frac{(x-t)^\lambda}{(y+\eta-\tau)^{1+\beta-\gamma}} \left[\frac{1}{(x-t)^{1+\alpha}} - \frac{1}{(x+h-t)^{1+\alpha}} \right] dtd\tau + \\
& + \int_0^x \int_0^y (x-t)^\lambda (y-\tau)^\gamma \left[\frac{1}{(x-t)^{1+\alpha}} - \frac{1}{(x+h-t)^{1+\alpha}} \right] \left[\frac{1}{(y-\tau)^{1+\beta}} - \frac{1}{(y+\eta-\tau)^{1+\beta}} \right] dtd\tau.
\end{aligned}$$

After which every term is estimated in the standard way, and we get

$$\left| \left(\Delta_{h,\eta}^{1,1} \varphi \right) (x, y) \right| \leq C_3 h^\lambda \eta^\gamma.$$

This completes the proof.

Main theorem. The mixed fractional integral operator $I_{0+,0+}^{\alpha,\beta}$ isomorphically maps the space $\tilde{H}_0^{\lambda,\gamma}(Q)$, $0 \leq \lambda, \gamma \leq 1$ onto the space $\tilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$, if $\lambda + \alpha < 1$ and $\gamma + \beta < 1$.

Proof. We should consider, as usual the following three parts of the proof:

- 1) Action of the mixed fractional integral operator from the space $\tilde{H}_0^{\lambda,\gamma}(Q)$ to the space $\tilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$;
- 2) Action of the mixed fractional differentiation operator from the space $\tilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$ to the space $\tilde{H}_0^{\lambda,\gamma}(Q)$;
- 3) The possibility to represent any function $f(x, y) \in \tilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$ as $(I_{0+,+}^{\alpha,\beta} \varphi)(x, y)$ with the density in $\tilde{H}_0^{\lambda,\gamma}(Q)$.

Because of (1) the parts 1) -2) are covered by Theorems 3 and 6. The part 3) is treated in the standart way in case $0 < \alpha < 1$ and $0 < \beta < 1$ by using the possibility of similar representation with the density from $L_p(\mathbb{R}^2)$, $\bar{p} = (p_1, p_2)$. See [1] Theorem 24.4.

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